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INVESTIGATION OF THE $He^3 + H^3$ *REACTION*

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Using a thin gas target, the total cross section for the $He^3 + H^3$ reaction was determined over the range of triton energies from 150 to 970 kev. In this energy interval, the total cross section increases monotonically from 3.2 to 63.0 mb. The energy dependence of the ratios of cross sections for different branches of the reaction were measured. We determined the energy of the decay of He^5 into an α particle and a neutron.

THERE have been just two experimental papers on the reaction

$$He^3 + H^3 \rightarrow Li^6 \rightarrow He^4 + H^2 + 14.31\ \text{Mev}, \qquad (1)$$

$$\rightarrow He^4 + H^1 + n + 12.08\ \text{Mev}, \qquad (2)$$

$$\rightarrow He^5 + H^1 + Q$$
$$\downarrow$$
$$He^4 + n + Q'. \qquad (3)$$

The work of Almqvist et al.[1] is qualitative. In the work of Moak,[2] which was carried out on a solid lithium target, values were obtained for the total cross section of the $He^3 + H^3$ reaction and for the ratio of the cross sections for various branches of the reaction, and the energy of decay of He^5 into an α particle and a neutron was determined. The precision in the determination of the total cross section is estimated to be ±20%.

The present work, unlike these preceding efforts, was carried out on a thin gas target. The total cross section for the reaction was determined by the method of integral counting of neutrons from the branches (2) and (3) of the reaction, taking account of the ratio between the various reaction branches. This ratio was found from the spectra of charged particles measured at an angle of 90° in the laboratory system. On the basis of the measured spectra of reaction products, an estimate was made of the energy of decay of He^5 into an α particle and a neutron. As a control experiment, we measured under the same conditions using the integral method the cross section for the reaction $H^2 + H^3$.

EXPERIMENTAL METHOD

Tritium ions accelerated by an electrostatic generator, after passing through a series of limiting diaphragms, entered the entrance window of the gas target. The thin gas target was placed in a vacuum chamber located in the center of a tank containing a 2% aqueous solution of $KMnO_4$. At an angle of 90° to the direction of the triton beam we placed a FÉU-25 photomultiplier with a CsI crystal, which served as a monitor to check the constancy of the yield of reaction products. The entrance window of the gas target was covered by a nickel foil $0.9 - 1.4\ \text{mg/cm}^2$ thick. The side window of the target which faced the scintillation counter was covered by a nickel foil $1\ \text{mg/cm}^2$ thick. The energy dependence of the energy loss of the tritons in the foils was determined by using an auxiliary magnetic analyzer.[3] A thermocouple was inserted in the gas target for determining its temperature. The target was filled with a 100% concentration of He^3 to a pressure of 60 mm of Hg.

The energy loss of the tritons in the gas target filled with He^3 was determined from the computed curve.[4] The flux of neutrons resulting from the

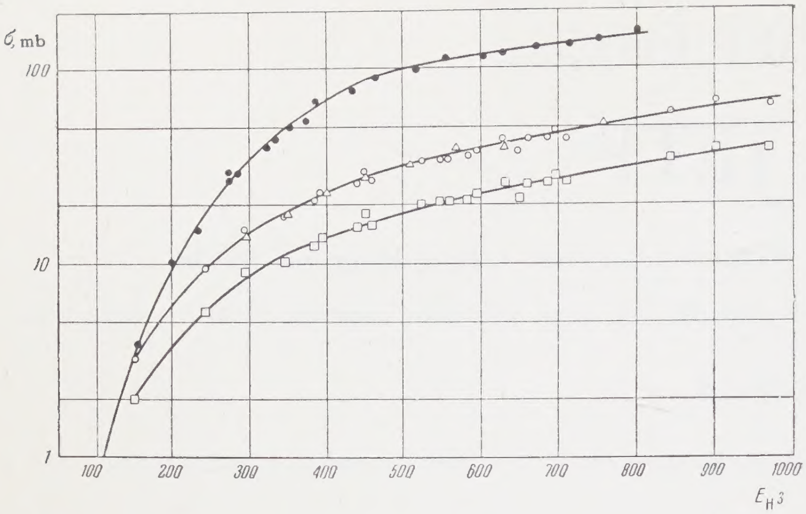

FIG. 1. Cross sections for the $He^3 + H^3$ reaction as a function of triton energy E_{H^3}; ● – total cross sections after Moak[2]; □ – cross sections for neutron branches; ○ – total cross sections obtained from the cross sections for the neutron branches and the ratio between the various reaction branches; △ – total cross sections obtained by counting charged particles.

interaction of H^3 and He^3 nuclei was determined in relative units by measuring the β activity induced in the manganese in the $KMnO_4$ solution. The absolute value of the neutron flux from the target was obtained by comparing it with the flux from a standard Ra + Be source in the same solution. To determine the ratio between the various reaction branches and also to measure the differential cross section we used a spectrometer consisting of a FÉU-S photomultiplier and a CsI crystal. The crystal was 20 mm in diameter and 1.8 mm thick. The spectrometer was placed at an angle of 90° with respect to the beam of accelerated tritons. In measuring the energy spectra, pulses from the photomultiplier passed through a cathode follower and amplifier, and entered a 50-channel pulse analyzer. The whole system was calibrated in energy and had excellent linearity.

RESULTS OF THE MEASUREMENTS

Figure 1 shows the results of measurements of cross sections for the neutron branches of the reaction, obtained by integral counting of neutrons.[5,6] The root mean square error of the measurements is ±5% in the energy range 240 – 970 kev, and ±31% at 149 kev.

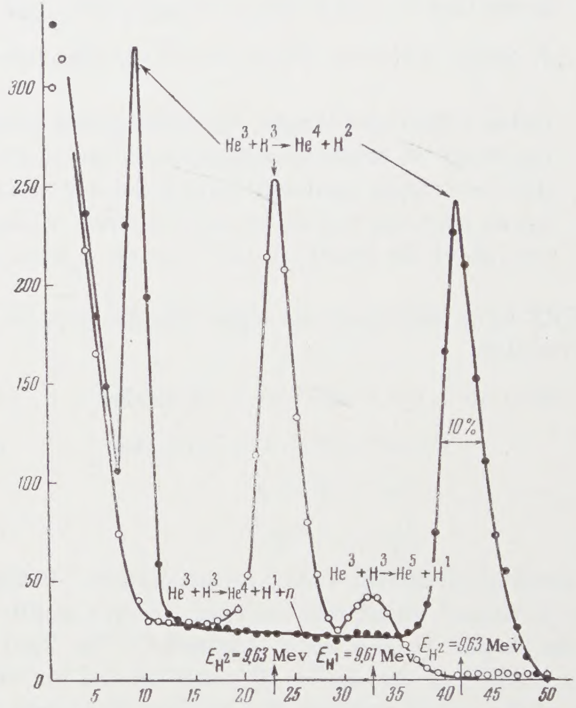

FIG. 2. Energy spectrum of particles from the $He^3 + H^3$ reaction for E_{H^3} = 506 kev: ● – total spectrum; ○ – the same spectrum after passing through an aluminum foil 59.08 mg/cm² thick. The abscissa gives the number of the channel in the analyzer, and the ordinate the relative number of counts per unit energy range (arb. units).

The ratios of the various branches for the $He^3 + H^3$ reaction (as a function of the energy of the incident tritons) was determined from measurements of the energy spectrum of charged particles at an angle of 90° in the laboratory system. One such spectrum is shown in Fig. 2. The spectrum of particles has two peaks, due to the α particles and deuterons of branch (1). Between these peaks there is a continuous spectrum of protons from branch (2). The proton peak, corresponding to the ground state of He^5, is close in energy to the deuteron peak from branch (1), so that it is not resolved by the spectrometer. The resolution needed for separating these peaks can be obtained by making use of the difference in range of protons and deuterons in any material. For this purpose we placed along the path of the particles from the gas target to the scintillation counter an aluminum foil 59.08 mg/cm² in thickness. After passing through this foil, the particles had quite different energies (for example, the protons had 7.37 Mev and the deuterons 5.48 Mev for $E_{H^3} = 636$ kev) and could easily be resolved. In Fig. 2 we show one of the spectra obtained using the foil. Similar spectra were taken for eleven values of the triton bombarding energy in the interval 150 – 950 kev. As the measurements showed, in the range of energies investigated the ratios of the various branches of the reaction $He^3 + H^3$ remain constant and are equal, on the average, to

$$He^3 + H^3 \to He^4 + H^2 + Q_1 \ldots\ldots (41 \pm 2)\,\%, \qquad (1)$$

$$\to He^4 + H^1 + n + Q_2 \ldots (55 \pm 2)\,\%, \qquad (2)$$

$$\to He^5 + H^1 + Q_3 \ldots\ldots (4 \pm 1)\,\% \qquad (3)$$

Using the cross sections obtained for the neutron branches (2) and (3), and the ratios of the contributions from the different branches, under the assumption of spherical symmetry of the angular distributions of the reaction products, we calculated the total cross sections. The computational results are shown in Fig. 1. On this same figure we show the total reaction cross section determined by counting of charged particles. These were obtained from the measured differential cross section at 90° by multiplying by 4π. Since the beam of tritium ions also contains some deuterium ions (the impurity amounts to 0.5 – 0.6%), in measuring the yield of charged particles we eliminated the contribution from the $He^3 + H^2 \to He^4 + H^1$ reaction. The root mean square error of the measurement of differential cross sections was ±6% on the average.

As a check on the operation of the whole apparatus, we made a measurement of the cross section for the $H^2 + H^3$ reaction by the integral method. The cross section for this reaction is known to good accuracy (3 – 5%) over a wide energy range, and, naturally, the agreement of the results obtained by our method with the data in the literature is a convincing proof of its suitability for determining the cross sections of the neutron branches of the $He^3 + H^3$ reaction. As our measurements showed,[7] the results obtained by us agree within the limits of error (±8%) with all the main papers concerning this reaction.

DETERMINATION OF ENERGY OF DECAY OF He^5 AND DISCUSSION OF RESULTS OF THE EXPERIMENT

From the experimentally determined energy of the protons from branch (3) we calculated the energy ϵ of decay of He^5 into a neutron and an α particle. The calculation was made using the formula

$$\varepsilon(He^5) = 0.4E_{H^3} - 1.2E_{H^1} + 12.08 \text{ Mev},$$

where E_{H^3} is the energy of the incident tritons; $E_{H^1} = (9.6 \pm 0.1)$ Mev is the energy of the protons from branch (3) (also in the lab system). The value found for the decay energy of He^5 is (0.8 ± 0.1) Mev.

The total cross sections given in Fig. 1 for the $He^3 + H^3$ reaction, as measured by two independent methods, agree with one another within the limits of error. On the same figure, we also show the results of Moak.[2] The cross sections found by Moak are three times higher than those of the present work. Such a large discrepancy is rather difficult to explain. We feel that the agreement of the values for total cross sections obtained in the present work by two independent methods, as well as the positive result of the control experiment for determining the cross section of the $H^2 + H^3$ reaction, speak convincingly in favor of the reliability of the values obtained by us for the total cross sections.

The data of the present work with regard to the ratio of the various branches of the $He^3 + H^3$ reaction are in good agreement with the data of Moak.

As for the decay energy of He^5, the value (0.8 ± 0.1) Mev obtained here is close to the value of $\epsilon(He^5)$ given in the summary of Ajzenberg and Lauritsen[8] which is (1 ± 0.1) Mev, and the value of (0.95 ± 0.07) Mev given by Moak.[2]

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INVESTIGATION OF SECONDARY (α, xn) REACTIONS IN BISMUTH

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The absolute production cross sections and relative yields of At^{210} and At^{211} from bismuth irradiated by 120 — 660 Mev protons were determined by radiochemical means. In the given case the astatine isotopes are produced in the (α, xn) reaction when an α particle with an energy $E_\alpha > 20$ Mev, produced as a result of profound disintegration, is captured by a target nucleus. The cross sections for the production of α particles with energies $E_\alpha > 20$ Mev were calculated on the basis of the experimental data, and the shape of their energy spectrum was estimated.

INTRODUCTION

THE disintegration of complex nuclei by high-energy protons produces alpha particles, some of which possess a sufficiently high energy to give rise to secondary reactions in the target nuclei.[1-4] Determination of secondary-reaction yields enables one to reach conclusions about the shape of the spectrum of fast α particles, and to estimate the cross section for their production. In particular, this method was employed to determine the shape of the spectrum and the production cross section of fast α particles produced in the decay of bismuth[1] and gold.[2]

Despite the fact that in both references single isotopes and elements with near atomic numbers were studied, the shapes of the spectra of the fast alpha particles differ from each other considerably. In addition, the cross section for the production of particles with $E_\alpha > 20$ Mev in bismuth turned out to be 5 to 6.3×10^{-25} cm², which is more by a factor of 3 — 4 than the cross section obtained from the data of Perfilov and Ostroumov[5] who used the method of three photoemulsion layers.

In view of these differences, we thought it useful to repeat the work of Kurchatov et al.[1] and to clarify the reason for the observed discrepancies.

EXPERIMENTAL PART

We conducted our work with bismuth of high purity, the total impurity content of which did not exceed 10^{-4}%. The targets were irradiated in the synchrocyclotron of the Laboratory for Nuclear Problems at the Joint Institute for Nuclear Research. The proton energy was varied between 120 and 660 Mev by placing the target at different orbit radii of the accelerated protons inside the vacuum chamber of the accelerator.

To prevent losses of astatine due to heating of the target during irradiation, the bismuth was sealed in a quartz ampule with an outer diameter of (3.9 ± 0.1) mm, a wall thickness of 0.5 — 0.6, and a length of 30 mm. The metal occupied approximately half the volume of the ampule. The time of irradiation varied from 5 to 15 min. The proton-beam intensity was estimated from the production of Na^{24} in aluminum foil which was wrapped about the lower half of the ampule occupied by the metal.

Three hours after the irradiation, the astatine was extracted from the bismuth and precipitated together with elementary tellurium.[6] A suspension of elementary tellurium in water was uniformly deposited on a stainless-steel disc with a diameter of 2 cm and was dried at a temperature of 80 — 90°C. To prevent mechanical losses of tellurium during the measurements, the preparation was covered with a film of nitrocellulose (0.02 mg/cm²).

The absorption of α particles in the tellurium layer and the covering film was determined experimentally. For this purpose we compared the α activity of equal amounts of astatine precipitated together with tellurium and covered with a film, and astatine deposited from a nitric-acid solution on a polished platinum disc. In the latter case the astatine preparation was without a carrier and there was no absorption of α particles in the layer. The experiment showed that the tellurium layer and the film which covered it absorbed 25%

of the α particles from At^{211} ($E_\alpha = 5.86$ Mev) and Po^{211} (7.44 Mev), and 30% of the α particles from Po^{210} (5.3 Mev).

The alpha activity of tellurium preparations which contained astatine was measured with the aid of a scintillation counter consisting of an FÉU-19 photomultiplier with a ZnS(Ag) scintillator. The natural background of the measuring apparatus amounted to 10 — 20 pulses/sec. The total efficiency determined with the aid of an α standard (Pu^{239}, $E_\alpha = 5.15$ Mev) was found to be ~ 35%. It was observed in the measurements that the activity of all preparations decreased with two half-lives: (7.3 ± 0.2) hr and 140 days; these are the half-lives of At^{211} and Po^{210}, respectively. Po^{210} is formed in the decay of At^{210} ($T_{1/2} = 8.3$ hrs; K capture).

The activity of the monitor specimens (Na^{24}) was measured with a end-window MST-40 Geiger-Mueller counter. The total efficiency, determined with the $Sr^{89}\beta$ standard ($T_{1/2} = 51$ days, $E_\alpha = 1.46$ Mev), amounted to ~ 20%.

RESULTS

The absolute cross sections for the production of At^{210} and At^{211} for various energies of the bombarding protons E_p are listed in the table. The absolute cross section for the production of astatine isotopes from bismuth under the action of fast protons is here understood to be*

E_p, Mev	$\sigma(At^{211})$, 10^{-29} cm²	$\sigma(At^{210})$, 10^{-29} cm²	$\sigma(At^{210})/\sigma(At^{211})$
130	0.52	0.33	0.63
170*	1.28	1.05	0.8
300	1.96	1.18	0.60
400	1.92	1.58	0.82
480*	2.52	2.10	0.8
530	2.82	2.28	0.81
580	2.26	1.67	0.66
660	2.60	2.14	0.82

*Corrected results of reference 1.

$$\sigma^{p}_{At} = n \int\limits_{20\ \text{Mev}}^{\infty} \sigma_\alpha dE \int\limits_{0}^{\infty} \frac{\sigma^{\alpha}_{At}}{-dE/dx} dE,$$

where n is the number of bismuth nuclei per cubic centimeter, σ_α is the cross section for the production of α particles on the target nuclei under the action of protons, σ^{α}_{At} is the cross section for the production of astatine isotopes under the action of α particles, and dE/dx is the energy loss of the α particles in the target material.

The ratio of the production cross sections of At^{210} and At^{211} is listed in the last column of the table. The production cross sections of the individual astatine isotopes are determined within a precision of ±30%, the error in the determination of their ratio does not exceed ±15%.

As can be seen from the results listed in the table, the production cross section of astatine isotopes increases by about a factor of five when the energy of the bombarding protons E_p is increased from 120 to 600 Mev. The cross section increases most rapidly up to energies of 300 — 400 Mev. The ratio of the cross sections $\sigma_{At^{210}}/\sigma_{At^{211}}$ as a function of E_p remains constant within the limits of the experimental error. This indicates that the shape of the spectrum of the α particles produced in the disintegration of bismuth remains practically unchanged in the 120 — 600 Mev range of proton energies.

The larger values of the production cross sections of At^{211}, and consequently also of the cross sections $\sigma_\alpha(E_\alpha > 20$ Mev), obtained in reference 1, are explained by the fact that in the processing of the experimental data no allowance was made for the registration of At^{211} from the α activity of the daughter element Po^{211} ($T_{1/2} = 0.52$ sec, $E_\alpha = 7.44$ Mev). If the results of this work are suitably corrected, the obtained production cross sections of At^{211} are in good agreement with our data (cf. the table).

Comparison of our data on the cross sections of the secondary reaction products for $E_p = 400$ Mev with those of reference 2 for gold ($E_p = 380$ Mev) shows that the production cross sections of products of the secondary (α, 2n) reaction are in satisfactory agreement with each other; 1.92×10^{-29} cm² for bismuth, and $(2.25 \pm 0.5) \times 10^{-29}$ cm² for gold. The cross sections of the products of the (α, 3n) reactions differ, however, markedly, amounting to 1.58×10^{-29} cm² for bismuth and $(4.2 \pm 0.8) \times 10^{-29}$ cm² for gold.

From our experimental data we calculated the cross sections for the production of α particles with an energy greater than 20 Mev in bismuth, and the shape of their spectrum. The following results were obtained:

E_p, Mev	130	170*	300	400	480*	530	580	660
$\sigma(E_\alpha > 20$ Mev), 10^{-25}cm²	0.42	1.03	1.58	1.55	2.03	2.28	1.82	2.1

The method used in the calculation is described in detail in reference 1. The α-particle ranges in the material for energy losses between E + ΔE and E were calculated with the aid of the well-known formulas.[7] The excitation functions of the (α, 2n) and (α, 3n) reactions were taken from the literature (cf. references 8 — 11). The decreasing portion of the excitation function of the (α, 3n) reaction in the 43 — 60 Mev range was calculated from the formulas of reference 12. The nuclear temperature was taken to be 1.5 Mev, and the neutron binding energy was calculated from the tables in references 13 and 14.*

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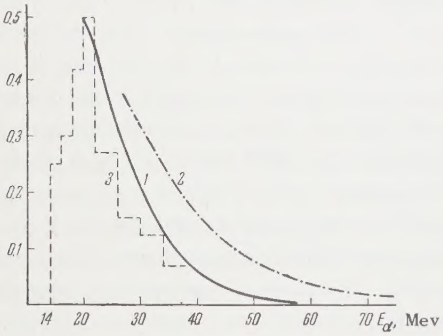
*Starting from the alpha spectrum, one can show that in the irradiation of thick specimens ($\delta > 1$ mm) σ^{p}_{At} is independent of the target thickness.

*Corrected results of reference 1.

The shape of the alpha spectrum was taken in the form

$$P(E) = \frac{E-V}{\tau^2} \exp\left(-\frac{E-V}{\tau}\right). \qquad (1)$$

The values of the parameters V and τ were chosen to satisfy the experimentally observed mean value of the ratio of the cross sections $\sigma_{\mathrm{At}^{210}}/\sigma_{\mathrm{At}^{211}} = 0.74$ in the falling form of the spectrum. Such values of the parameter are $\tau = 6$ Mev and V = 12 Mev. The spectrum calculated with these values of τ and V is shown in the figure.



The spectra of the fast α particles (in relative units): 1 — the spectrum calculated from the At^{210} and At^{211} yields from bismuth; 2 — the spectrum calculated from the yields of the thallium isotopes from gold (reference 2); 3 — averaged spectrum of α particles emitted by Bi and W (reference 5).

The same figure shows the spectrum of the α particles produced in the decay of bismuth and tungsten,[5] and the alpha spectrum calculated from the experimental data on secondary reactions in gold.[2] The curves are normalized in such a way that the areas under the curves are proportional to the production cross sections of α particles with corresponding energies.

Our attention is drawn by the difference in the energy spectra of α particles calculated from the secondary-reaction yields in bismuth and gold. In the latter case the spectrum has a smaller slope, and the cross section for the production of α particles turns out to be larger than in bismuth. It seems to us that this discrepancy is due to a possible error in the determination of the relative yields of thallium isotopes because of the difficulty in analyzing the total decay curve of these nuclei. It is impossible to explain this observed discrepancy from the features of the excitation function of the (α, xn) reactions in gold, since a calculation we made with the aid of the formulas of reference 12 showed that the excitation functions for gold are almost the same as for bismuth. However, the authors of reference 2 take account of α particles starting from 27 Mev while the experimental excitation function of the (α, 2n) reaction in bismuth and the calculated excitation function in gold begin at an α-particle energy of ~ 20 Mev.

The production of α particles with an energy larger than 20 Mev cannot be exclusively explained by an evaporation mechanism. Actually, an alpha spectrum satisfying the experimentally obtained ratio of the production cross sections of the two astatine isotopes could only be explained by the evaporation of α particles at an excitation energy of the bismuth nucleus of ~ 600 Mev. Such an excitation energy is unlikely for 600-Mev protons, and completely excluded for protons with lower energies.

On the other hand, the evaporation spectra of α particles with an energy larger than 20 Mev calculated for mean excitation energies of the bismuth nucleus (E_{exc}) due to 120- and 660-Mev protons fall more steeply than was experimentally observed. The spectra were calculated according to formula (1) with $\tau = (cE_{\mathrm{exc}})^{1/2}$ (c = 12.4/A, A is the atomic weight).[16] The lowering of the Coulomb barrier was taken account of both by means of the formula[16] $V = 0.83V_{\mathrm{class}}/(1 + E_{\mathrm{exc}}/200)$, and by means of the formula $V = 0.83V_{\mathrm{class}}(1 - \tau^2/\tau_c^2)$, where $\tau_c = 9$ Mev.[17] The mean excitation energy of the bismuth nucleus was taken (on the basis of the data of reference 18) to be 190 and 80 Mev for proton energies of 660 and 120 Mev, respectively.

Thus, even at the maximum excitation energies possible in our instance, the obtained spectrum cannot be explained by the evaporation theory.

The fact that the shape of the energy spectrum of the α particles produced in the decay of bismuth nuclei, within the sensitivity of our experiment, does not depend on the energy of the bombarding protons also speaks in favor of this conclusion.

If it is assumed that all 20-Mev α particles are due to evaporation, then it follows from the calculations that on the average half of the α particles with $E_\alpha > 20$ Mev are produced by a different mechanism than the evaporation of α particles from a homogeneously hot excited nucleus. We assume that the fast α particles can be reasonably assumed to have been knocked out from the peripheral region of the nucleus in the process of an intranuclear cascade.[19]

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*Cross sections for the capture of α particles by the Bi nucleus were calculated with the aid of a formula proposed by Babykov in reference 15.

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MAGNETOCALORIC EFFECT IN THE RANGE OF THE LOW-TEMPERATURE TRANSFORMATION OF MAGNETITE

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The temperature dependence of the reversible magnetocaloric effect was studied on magnetic single crystals between 80 and 290°K. Two minima have been discovered and are attributed to a change in the anisotropic properties of the crystal. The effect of the conditions of cooling of the magnetite on the magnitude of the magnetocaloric effect has been studied. An irreversible change of the temperature in a relatively narrow temperature range (95 — 115°K) has also been detected.

INTRODUCTION

As is well known, magnetite (Fe_3O_4) has a cubic structure of the inverted-spinel type. At temperatures close to 110°K, a transformation takes place in magnetite and leads to a sharp anomalous change in its electric,[1-3] magnetic,[3,4] galvano- and thermomagnetic,[5,6] and also thermal[7] properties. As a result of this transformation, the magnetic symmetry also changes from cubic to uniaxial.[8] Lee[8] has shown that if the single crystal of magnetite is cooled in a sufficiently strong magnetic field, then one of the axes of the cube, closest to the direction of the external field, becomes the direction of easy magnetization or the symmetry axis at temperatures below the transformation temperature. On going through the transformation region, the magnetization in weak fields changes radically, while the saturation magnetization remains practically unchanged.

The nature of this low-temperature transformation of magnetite is still not fully explained. Verwey et al.[1] have shown that the transformation is due to the ordering of divalent and trivalent iron ions over the octahedral lattice sites. This ordering is due to the change in the structure, in which the crystal symmetry goes from cubic at temperatures above the transformation into orthorhombic.[9] Bickford[10] investigated the temperature dependence of the anisotropic constant of a single crystal of magnetite, starting with room temperatures and below, and found that the anisotropy constant increases sharply in the region of the low-temperature transformation. As follows from the theory,[11] in the temperature region where the anisotropy constant of the crystal experiences a sharp change with temperature, one can expect a large reversible magnetocaloric effect in the region of the rotation fields. On the other hand, an investigation of the magnetocaloric effect permits a qualitative study of the behavior of the anisotropic constant.

The purpose of this investigation was to ascertain the singularities in the behavior of the magnetocaloric effect in the region of the low-temperature transformation of magnetite, and also to investigate qualitatively the temperature dependence of the first anisotropic oonstant in this temperature region.

As is well known, adiabatic magnetization is always accompanied by a change in the temperature of the magnetized body, the value of which can in general be represented in the form

$$\Delta T = \Delta T_1 + \Delta T_2 + \Delta T_3 + \Delta T_4, \qquad (1)$$

where ΔT_1, ΔT_2, and ΔT_3 are the changes in temperature due respectively to the reversible processes of displacement, rotation, and true magnetization, while ΔT_4 is the temperature change due to all other irreversible processes. The contribution made by each term in (1) depends, in particular, on the temperature and on the applied magnetic field. In the region of sufficiently low temperature ($T \ll \Theta_f$, where Θ_f is the Curie temperature), the value of ΔT_3 is relatively small for magnets in which the susceptibility of the para process tends to zero as $T \rightarrow 0$°K. In the region of rotation fields $H < H_s$ (H_s is the saturation field), in the absence of irreversible processes, the main contribution to ΔT is made by ΔT_2, i.e., the change in temperature is determined essentially by the magnetic energy of the crystallographic anisotropy.

It follows from the work of Akulov et al.[11] that in this case ΔT is connected with the derivative of the anisotropy constants with respect to the temperature, $\partial K/\partial T$, by the following relation (we discard terms containing $\partial^2 K/\partial T^2$ and derivatives of higher order):

$$\Delta T = -(T/C_{pI})(\partial K/\partial T)_H \Delta \sum_{i \neq j} \alpha_i^2 \alpha_j^2, \qquad (2)$$

where C_{pI} is the specific heat at constant p and magnetization I, while α_i and α_j are the direction cosines of the spontaneous magnetization relative to the cubic axes of the crystal. We can thus assume that, subject to several conditions, the experimentally-measured magnetocaloric effect ΔT is essentially determined by (2), and that the magnitude of the effect is a function of both the temperature and the intensity of the applied magnetic field. If the measurements are carried out at constant fields, the value of ΔT is proportional to the derivative $\partial K/\partial T$ for the corresponding temperature, and this makes possible a qualitative investigation of the temperature variation of the anisotropy constant.

MEASUREMENT SETUP AND PROCEDURE

We investigated two natural single crystals of magnetite of different stoichiometric composition. The samples were in the form of spheres, 9.50 ± 0.5 (sample 1) and 10.64 ± 0.2 (sample 2) mm in diameter. The measurements of the magnetocaloric effect for sample 1 was with a potentiometer circuit of sensitivity 5×10^{-8} v/mm, corresponding to a change of 3×10^{-3} deg in the sample temperature. Sample 2 was investigated with the aid of another potentiometer circuit, including a dc photoelectron-optical amplifier (type FÉOU-18) and a ÉPP-09 automatic recorder. The sensitivity of the second setup was 10^{-8} v/mm, corresponding to a temperature change of approximately 1×10^{-3} deg. The measuring element was a copper-constantan thermocouple, soldered to the surface of sample 1 or inserted inside sample 2. To reduce heat losses, the thermocouple wires were made of the smallest diameter possible without allowing the wire resistance to decrease greatly the sensitivity of the apparatus.

The sample was placed inside a silvered Dewar vessel 20 mm in diameter, the open end of which was immersed in a second Dewar of larger diameter.[6] The sample temperature was varied by means of a bifilar heating coil located inside the inverted Dewar. The coil was fed with alternating current through a ferroresonant voltage stabilizer. A cryostat of this kind maintains the temperature stable within ± 10^{-3} deg over a time sufficient to perform a measurement cycle.

The experimental difficulties with measuring small values of adiabatic temperature changes in ferromagnetic semiconductors of low thermal conductivity have been described, in particular, by Clark and Sucksmith.[12] In our procedure these difficulties were coped with by measuring ΔT in several samples with different thermocouple locations, by choosing thermocouple wires of different thickness, and also by using a Dewar with reflecting walls.

After the equilibrium temperature of the sample was established, the electromagnet was turned on and the change in the sample temperature was determined from the deflection of the galvanometer. After 20 minutes, the sample again assumed its initial temperature and the electromagnet was turned off. A temperature change in the opposite direction was then observed. The absolute changes in the sample temperature due to turning the magnetic field on and off were equal, within the limits of experimental accuracy, and their average was taken as the value of the magnetocaloric effect ΔT. To eliminate the ballistic overshoot of the galvanometer coil, a compensating coil with a small number of turns was connected in the thermocouple circuit.

The same procedure was used to investigate the temperature dependence of the magnetocaloric effect in both samples. The sample was first demagnetized at room temperature and then cooled to the boiling temperature of liquid nitrogen in the absence of a magnetic field. The temperature dependence of the magnitude of the effect was investigated as the sample was heated. This was followed by an investigation of the influence of conditions of cooling in a magnetic field of different intensity on the magnitude of the effect. The sample, previously demagnetized at room temperature, was cooled in a magnetic field of given intensity H_0 to the boiling temperature of liquid nitrogen, and then the temperature was raised to a value somewhat lower than the transformation temperature. The dependence of the magnitude of the effect ΔT on the intensity of the magnetic field was plotted in the transformation region, the measurement being started with the lowest value of the field and terminated with the largest. The sample was then heated to room temperature, demagnetized, and cooled in a magnetic field of different intensity.

We know that magnetite displays a jump in magnetization in weak magnetic fields, and consequently this effect must be investigated in fields strong enough to produce saturation. Our measurements of the temperature dependence of the magnetization I of the investigated specimens in the interval from 85 to 125°K have shown that a plot of I(T) measured in a field of 15,000 oe, no longer has a jump. The direction of the magnetic field in all the measurements coincided with the crystallographic [111] direction of the single crystal. The absolute error in the measurements of ΔT did not exceed 2×10^{-3} deg in the entire temperature interval.

MEASUREMENT RESULTS AND THEIR DISCUSSION

Figure 1 shows the temperature dependence of the magnetocaloric effect ΔT for samples 1 and 2. Both curves show the course of heating of samples first demagnetized at room temperature and then cooled to 80°K in the absence of a magnetic field. The curves for samples 1 and 2 were obtained in magnetic fields of 13,800 and 17,000 oe, respectively. As can be seen from the figure, a reversible temperature change of opposite sign takes place at low temperatures, i.e., the sample temperature decreases as the field is turned on. The effect vanishes at T = 125°K for sample 2 and T = 145°K for sample 1. Further increase in the temperature makes the effect positive. A striking fact is the presence of a sharp minimum of the curve at T = 104° for sample 1 ($|\Delta T|_{\max} = 0.100°$) and at T = 110°K for sample 2 ($|\Delta T|_{\max} = 0.105$).

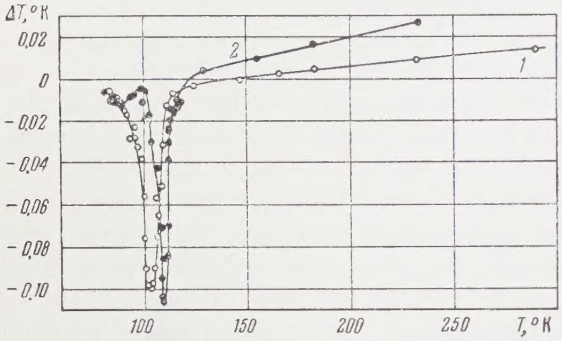

FIG. 1. Temperature dependence of the magnetocaloric effect for samples 1 (curve 1) (H = 13,800 oe) and 2 (curve 2) (H = 17,000 oe).

It must be noted that a "first measurement effect" is observed in the region of the minimum of the curves. If the specimen was kept at the measurement temperature in the absence of a magnetic field for approximately 20 minutes, the first turning on of the magnetic field led to an irreversible reduction in the temperature, the value of which was approximately double that obtained in the subsequent applications of the magnetic field. In all the subsequent applications, the magnitude of the magnetocaloric effect was always reproducible and fully reversible. The results of the measurements of the reversible change in temperature and of the "first-measurement effect" are shown in Fig. 2 for sample 1, cooled in the absence of a magnetic field. The same phenomenon was observed also for sample 2, but in the vicinity of 110°K. As can be seen from Fig. 2, the "first-measurement effect" (curve 2) reaches a minimum in the vicinity of 105°K, as does the reversible change in the temperature (curve 1). At this temperature the difference between these curves reaches a maximum value. On both sides of the minimum, the curves tend to come together and the "first-measurement effect" practically disappears at 95° and 115°K.

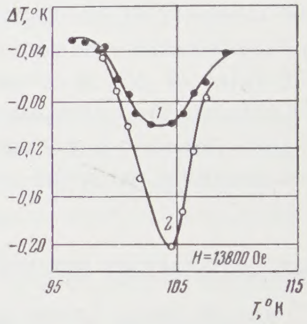

FIG. 2. Temperature dependence of the reversible magnetocaloric effect for sample 1 (curve 1) in the vicinity of the temperatures where the 'first-measurement effect" is observed (curve 2).

The dependence of the magnitude of the effect ΔT on the intensity of the magnetic field is shown for sample 2 in Fig. 3 (the curves were taken at T = 109.6°K). As can be seen from Fig. 3, in weak fields (up to 3,000 oe) the magnetocaloric effect is positive. The magnitude of this effect is relatively small (on the order of 0.01°) and reaches a maximum at the weakest field (to 1,000 oe), vanishing at approximately 3,000 oe. Above 3,000 oe, the effect is negative and increases with increasing field. Curve 2 of Fig. 3 shows the influence of cooling the sample on the magnitude of the effect. The sample was cooled to a temperature below that of the magnetic transformation in a field $H_0 = 17{,}000$ oe (parallel to the [111] crystal axis). As can be seen from a comparison of curves 1 and 2, cooling in a magnetic field reduces the effect substantially,

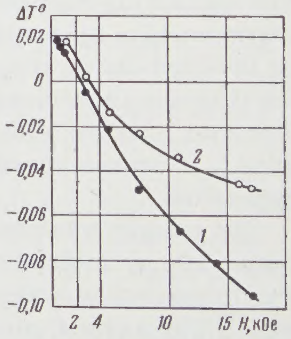

FIG. 3. Dependence of the magnetocaloric effect ΔT on the intensity of the magnetic field H for sample 2 at T = 109.6°K; curve 1 – sample cooled in the absence of magnetic field; curve 2 – sample cooled at H = 17,000 oe.

without influencing the dependence of the effect on the field. Cooling the sample in a magnetic field of lower intensity leads to analogous curves, located between curves 1 and 2. It must be indicated that as the sample is cooled in a magnetic field, the "first-measurement effect" is retained; all that is observed is a reduction in both the reversible and irreversible effects. The relation between the magnitudes of these effects remains the same. The value of ΔT for sample 1 depends analogously on H and on the method of cooling the sample with and without a field.

The reversible negative temperature change shown in Fig. 1 is undoubtedly connected in the main with the rotation processes. In this temperature region the para process has a relatively small influence. It makes a contribution on the order of + 0.01°, as estimated with the aid of the values of C_p and I_s taken from references 7 and 10. With increasing temperature, the influence of the para process increases relative to the rotation processes, as a result of which the magnetocaloric effect reverses sign and becomes positive. The presence of a sharp minimum on the temperature-dependence curves of the magnetocaloric effect is connected with the low-temperature transformation of magnetite, at which the crystallographic symmetry of the lattice changes. The curve taken for sample 2 has still another minimum, near 90°K, which is much smaller in magnitude than the first. No second minimum was observed for sample 1, (possibly because of the lower sensitivity of measuring apparatus), but a sufficient dispersion of the experimental curves was observed in the vicinity of 90°K. In the same temperature region, as already indicated,[6] one observes anomalies of the electric, galvanomagnetic, and thermomagnetic properties. We cannot exclude the possibility that the magnetite experiences a transformation, which has not yet been investigated, in this temperature region.

It seems to us that an unequivocal explanation of the difference between curves 1 and 2 of Fig. 2 is at present still impossible. We can only assume that the irreversible effect (curve 2) is connected with the occurrence of an orthorhombic structure in the magnetite. Narovskaya[13] has shown that the irreversible thermal losses, in the region of the magnetite transformation and below, may be due to a shift of the **c** axis of the orthorhombic crystal from its initial direction to a direction along the edge of the cube closest to the direction of the applied field. However, we did not observe this effect at temperatures below 95°K.

The character of the dependence of ΔT on the intensity of the magnetic field (Fig. 3) can be apparently attributed to a shift in the predominance of certain magnetization processes over others. At fields from 0 to 3,000 oe, where the effect is positive, the main contribution should be made by the irreversible displacements of the domain boundaries. It is possible here to correlate our measurements with the results of Bryukhatov,[14] who has shown, by measuring the magnetic moments in a rotating magnetic field, that in magnetite the hysteresis losses have a maximum at $H \approx 1{,}000$ oe and vanish at $H \approx 3{,}000$ oe. In fields of 3,000 oe and above, rotation processes predominate and lead to a reversible change in the temperature (except for the first measurement) with a negative sign. In this range of fields, the magnitude of the effect is in the first approximation linear in H^{-1}. The indicated character of the connection between ΔT and H indicates that the contribution of the displacement processes prevails over those of true magnetization, for according to theory, the magnetocaloric effect due to the para process should depend linearly on the intensity of the magnetic field in the region of low temperatures. In particular, as shown by Ivanovskiĭ,[15] the magnetocaloric effect for crystals with hexagonal symmetry depends linearly on H^{-1} in the range of rotation fields.

The influence of prior cooling of the sample in a magnetic field on the magnitude of the effect, shown in Fig. 3, can be apparently explained by the fact that cooling in a strong magnetic field leads to the appearance of an easy-magnetization axis, while cooling without a field makes the magnetite magnetically harder. The occurrence of an easy-magnetization axis causes the processes of rotation and orientation of the spontaneous magnetization along this axis to be accompanied by much smaller changes in the crystal temperature.

By using relation (2), it is possible to explain the temperature variation of the derivative of the anisotropy constant K and thereby obtain a qualitative idea of its variation with temperature. As was shown by Bickford,[10] only the first term (K_1) of the series expansion of the anisotropy energy need be considered, owing to the smallness of the remaining terms compared with K_1. The corresponding results for sample 2 are shown in Fig. 4, where the values of the specific heat are those given by Millar.[7]

It is seen from the figure that the derivative of the anisotropy constant with respect to temperature reaches its maximum value in the transformation region (110°K), and experiences thereby a sharp variation with temperature in a relatively small temperature interval (on the order of 15°). In the temperature region below the transformation point,

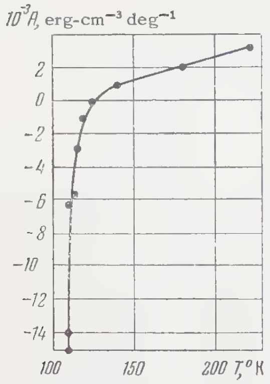

FIG. 4. Temperature dependence of the derivative of the anisotropy constant for sample 2. $A = (\partial K_1/\partial T)\Delta \sum_{i \neq j} \alpha_i \alpha_j^2$

the crystal lattice of the magnetite has no longer a cubic but a rhombohedral symmetry, and here it is necessary to take into account[16] the presence of several anisotropy constants, which are nearly equal in order of magnitude.

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ASYMMETRY OF $\pi^+ \rightarrow \mu^+ \rightarrow e^+$ DECAY IN A MAGNETIC FIELD

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The dependence of spatial asymmetry in $\pi^+ \rightarrow \mu^+ \rightarrow e^+$ decay on magnetic field strength has been measured in the range from 0 to 17 koe. It is shown that this dependence can be explained by the Paschen-Back effect for muonium upon the assumption of additional depolarization due to charge exchange toward the end of the μ^+ range or to electron exchange after stopping. For a NIKFI-R emulsion the minimum value of the asymmetry coefficient is -0.09 ± 0.01 (in zero field); the maximum value is -0.29 ± 0.01 (in strong fields). The 10% difference between -0.29 ± 0.01 and the value -0.33 predicted by the V-A theory greatly exceeds the experimental error. No three-electron decays were detected among the 340,000 $\pi^+ \rightarrow \mu - e$ decays examined.

1. INTRODUCTION

THE discovery of parity nonconservation in $\pi \rightarrow \mu \rightarrow e$ decay has led to the experimental study of the new hydrogen-like atom muonium, which is a stable system consisting of a μ^+ and an electron. This system can be formed when the velocity of a μ^+ as it slows down is comparable with the velocities of atomic electrons. With further slowing down the μ^+, like a slowed-down proton, repeatedly loses and captures an electron (charge exchange) and finally reaches thermal velocity in either a free or a bound state. A sufficiently slow μ^+ can no longer acquire or lose an electron through ionization; its interaction with electrons is then confined to electron exchange with atoms of the medium. The initial formation of muonium and its subsequent charge exchanges and exchanges with unpolarized electrons may account for the experimentally observed depolarization of μ^+ in various substances. For the purpose of testing these ideas experimentally we investigated the effect of a static magnetic field on the polarization of a μ^+; this had previously been studied in references 1—4. Since the suggested depolarization mechanism is the interaction between the μ^+ and electron magnetic moments in the muonium ground state (hyperfine splitting), a strong magnetic field which destroys the coupling between the two magnetic moments (the Paschen-Back effect) should also oppose depolarization and thus conserve the asymmetry of the angular distribution of positrons from $\mu^+ \rightarrow e^+$ decay. The quantum-mechanical theory of this effect[5] shows that enhanced polarization P with increasing field strength is represented by

$$P = 3a = \xi(0.5 + 0.5\, x^2/(1 + x^2)), \qquad (1)$$

where $x = H/H_0$ is the ratio of the applied magnetic field to the field $H_0 = 1580$ oe, which is the average field due to the μ^+ magnetic moment at the electron orbit in muonium, and ξ is an asymmetry parameter which represents the degree of longitudinal μ^+ polarization in $\pi^+ \rightarrow \mu^+$ decay and which is three times as large as the asymmetry coefficient a in $\pi \rightarrow \mu \rightarrow e$ decay.[6]

It follows from this formula that in fields of the order of tens of kilooersteds ($x^2 \gg 1$) there is practically no depolarization ($P \rightarrow \xi$), and that in weak magnetic fields ($x^2 \sim 0$) polarization should amount to half of its maximum value ξ.

The present work investigates the dependence of asymmetry in $\pi^+ \rightarrow \mu^+ \rightarrow e^+$ decays within NIKFI-R emulsion on magnetic fields ranging from 0 to 20 koe. We measured the asymmetry coefficients in emulsions which were carefully shielded ($H = 0$) and in fields of 54, 110, 206, 420, 680, 1300, 1900, 2500, 3500, 5100, 6300, 14,000 and 17,000 oe parallel to the plane of the emulsion. The emulsions were placed in a double soft-iron shield for protection from the magnetic field existing in the synchrocyclotron room. The vertical magnetic field at the center of the shielded space was $< 10^{-2}$ oe. Weak fields of 54—420 oe were excited by an air-core coil of 16 cm inside diameter and 60 cm length placed inside the double cylindrical magnetic shield. Fields of 1900—6300 oe were excited by a coil of 14 cm inside diameter surrounded by a heavy closed core of iron. The field within the emulsion was uniform to within 1%. Irradiation in fields of 14 and 17 koe required the use of a special electromagnet, for which we are indebted to I. I. Gurevich and B. A. Nikol'skiĭ.

2. IRRADIATION AND SCANNING

Pellicle stacks consisting of 50 to 100 NIKFI-R emulsions 400μ thick and measuring 10 × 10 or 10 × 15 cm^2 were irradiated in a π^+ beam produced by the synchrocyclotron of the Joint Institute for Nuclear Research. The absorbers which followed the collimator were calculated to cause stopping of the π^+ mesons in the middle of the pellicle stack.

In the case of a magnetic field it is more convenient to analyze the data with the usual angular correlation formula $dN \sim (1 + a \cos \vartheta)\, d\cos\vartheta$ replaced by

$$dN \sim (1 + a \cos\gamma \cos\beta)\, d\cos\gamma\, d\cos\beta, \qquad (2)$$

where γ and β are the angles formed by the μ^+ and electron paths, respectively, with the magnetic field direction.[4] Asymmetry was then measured as follows. A scale based on two mutually perpendicular lines was placed in the field of view of the micrometer eyepiece; the magnetic field direction bisected the right angle between the lines. The observers recorded the $\pi^+ \rightarrow \mu^+ \rightarrow e^+$ decays contained entirely in a given emulsion layer when the projections of the angles γ and β on the emulsion plane lay in the interval 0 ± 45° (first quadrant of the scale) or 180 ± 45° (third quadrant) while the vertex of the $\mu \rightarrow e$ decay was at least 50μ from the emulsion or glass surface. 99,442 $\pi^+ \rightarrow \mu^+ - e$ decays satisfying these criteria were registered in different magnetic fields. N_f will denote the number of decays for which the projections of γ and β lie in the same (first or third) quadrant of the ocular scale, while N_b denotes the number of decays for which the angular projections lie in opposite quadrants. The asymmetry coefficient is then given by

$$a = K(N_f - N_b)/(N_f + N_b).$$

K = 1.57 if we assume in first approximation, correct to within a few percent, that all μ^+ tracks lie in the plane of the emulsion while the electron tracks are distributed isotropically with respect to the vertical direction.

Table I gives the measured values of N_f and N_b and the corresponding calculated asymmetry coefficients a for the magnetic fields H that were used. The value K = 1.57 was subjected to a 3% correction to allow for the actual distributions of μ^+ tracks and of positron tracks around the vertical direction.[2]

TABLE I

H, oe	N_f	N_b	$N = N_f + N_b$	$-a$
$<10^{-2}$	7363	8270	15 633	0.093±0.013
54	3005	3477	6 482	0.117±0.020
110	3133	3592	6 725	0.110±0.020
206	2897	3247	6 144	0.093±0.020
420	2778	3317	6 095	0.142±0.020
680	3921	4718	8 639	0.149±0.017
1300	4052	4830	8 882	0.142±0.017
1900	2214	2692	4 906	0.155±0.022
2500	3246	4200	7 446	0.206±0.020
3500	2854	3589	6 443	0.185±0.020
5100	2619	3339	5 958	0.196±0.020
6300	2384	3319	5 703	0.265±0.021
14 000	1333	1848	3 181	0.262±0.028
17 000	2977	4228	7 205	0.282±0.018

Note. The value of a for $H < 10^{-2}$ oe was obtained in two runs, in the first of which the entire angular distribution was measured, while in the second run the measurements were performed as for $H \neq 0$. The given value is the weighted average for the two runs.

3. DISCUSSION

Table I and the figure show the values obtained for the asymmetry coefficient in different magnetic fields. We shall first consider the asymmetry in zero field and in very strong fields, both of which are especially important for an analysis of the data.

In zero field we obtained a(0) = −0.093 ± 0.013, which agrees with the value −0.092 ± 0.018 given in reference 1 for the same NIKFI-R emulsion.

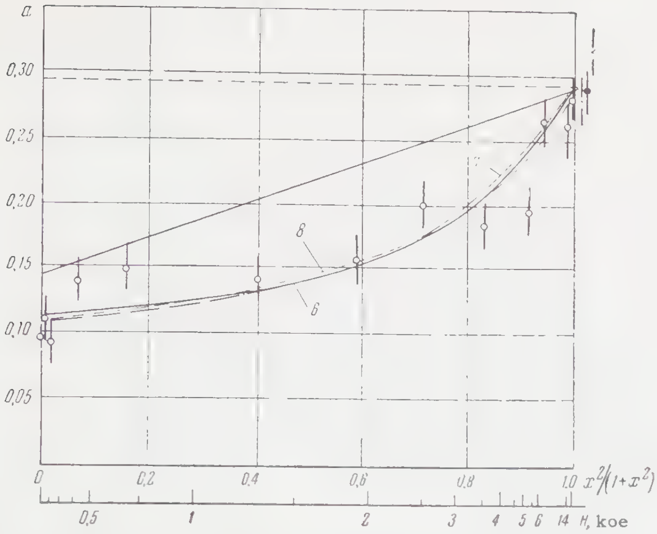
Our value for a maximum field of 17,000 oe is −0.282 ± 0.018. Table II shows our values and all values given in the literature for strong fields.

TABLE II

Source	H, koe	$-a$
Our data	17	0.282±0.018
Reference 7	25	0.290±0.013
Reference 8	20	0.280±0.020
	27	0.320±0.200

Combining these consistent data, we obtain a = −0.292 ± 0.009. It follows from (1) that the value of a in strong fields ($H \gg H_0$) gives directly ξ = 3a = 0.88 ± 0.03, which is denoted by a dashed line in the figure.

The straight line in the figure shows the dependence of a on $x^2/(1 + x^2)$ according to (1). The experimental data disagree completely with this formula and with the hypothesis that a increases linearly; therefore another depolarization mechanism must exist in addition to that described by (1). As noted above, this mechanism may be either charge exchange or electron exchange between muonium and the medium.



Dependence of asymmetry coefficient a on $x^2/(1 + x^2)$, where $x = H/H_0$. Curves 6, 7, and 8 were calculated by means of (6), (7) and (8); ○ – present data, ● – from reference 7, × – from reference 8. In order to avoid crowding the figure for $x \sim 1$, the values of a for H = 20, 25 and 27 koe are shown beyond the line drawn for $x^2/(1 + x^2) = 1$.

Since the electrons of the medium are unpolarized, every electron exchange entails further "dilution" of the remaining polarization, and a few exchanges are sufficient to greatly reduce or even eliminate μ^+ polarization in weak fields. The following polarization formula given by Sens et al.[9] is a simple extension of (1) to the case in which all n exchanges occur in a time that is longer than the hyperfine splitting time but considerably shorter than the μ^+ lifetime:

$$P = \xi\,[(1 + 2x^2)/2\,(1 + x^2)]^n. \tag{3}$$

When $n = 1$ this formula goes over into (1), and when $x^2 \to \infty$ we have $P \to \xi$ as in the case of (1).

A. M. Perelomov has communicated to us privately the following polarization formula, based on the assumption that electron exchanges occur throughout the entire μ^+ lifetime:

$$P = \xi\,(1 - {}^1\!/_2\Delta^2)\,(1 + W/\Delta^2)^{-1}, \tag{4}$$

where $\Delta^2 = 1 + x^2$ and W is the number of exchange collisions during the mean μ^+ lifetime. As in the case of (3) this formula goes over into (1) for $W = 0$ and polarization is entirely restored in strong fields.

Ferrell et al.[10] considered the additional depolarization mechanism to be charge exchange while an emitted μ^+ is still fast enough to lose and capture electrons several (n) times. If each capture lasts an average time τ equal to the mean muonium lifetime (in units of $\hbar/\Delta E = 3.6 \times 10^{-11}$ sec) their formula becomes

$$P = \xi\,(1 - 0.5\,/\,(1 + \tau^{-2} + x^2))^n \tag{5}$$

with limiting transitions similar to those of the preceding formulas.

The analysis of our data must take into account the fact that a fraction of the μ^+ stoppings occur in gelatin, where depolarization is either weak or absent, and the remainder in AgBr crystals, which are responsible for most of the depolarization.[11] Accordingly we have $f = (0.093 \pm 0.013)/(0.292 \pm 0.009) = 0.32$ as the ratio between the number of stoppings in gelatin and in AgBr. For the purpose of analyzing the data we obtain

$$P = 3a = \xi\,[f + (1 - f)\,(1 - 0.5/(1 + \tau^{-2} + x^2))^n]. \tag{6}$$

A similar analysis can be performed for the field dependence of the asymmetry coefficient that is represented by (3) and (4), which are based on the assumption of electron exchange as the cause of additional depolarization. The collision process is then characterized by the single parameter W [Eq. (7)] or n [Eq. (8)]. The second parameter that we can obtain in this case is $k = \rho\,(1 - f)$, where $(1 - f)$ is the relative number of stoppings in AgBr and ρ is the probability of muonium formation in these stoppings.

The formulas corresponding to (3) and (4) are

$$P = \xi\left[(1 - k) + k\left\{\frac{1 + 2x^2}{2\,(1 + x^2)}\right\}^n\right], \tag{7}$$

$$P = \xi\,[(1 - k) + k\,(1 - 1/2\Delta^2)\,(1 + W/\Delta^2)^{-1}]. \tag{8}$$

The method of least squares was used to compare the experimental data with (6), (7), and (8), and to determine the corresponding values of the parameters. The results are shown in Table III and in the figure. The first column of the table indicates the formulas used in the analysis, the second column gives the optimum values of the corresponding parameters, the third column gives the values of $\chi^2_{\min} = \Sigma N_i\,(\Delta a_i/1.57)^2$ corresponding to these parameters, and the fourth column gives the probabilities $p\,(\chi^2 > \chi^2_{\min})$.

TABLE III

Formula	Parameters	χ^2_{min}	p,%
(6)	$\tau = 1{,}3;\ n = 7$	21	7
(7)	$W = 3{,}3;\ k = 0.7$	18	15
(8)	$n = 6;\ k = 0.65$	18	15

Satisfactory agreement is found in all three instances. Thus a few charge exchanges in a path ending or a few electron exchanges after stopping can account for the measured field dependence of the asymmetry coefficient. The difference between the values of $\chi^2_{\min}$ is too small to permit any choice among the three possibilities.

4. CONCLUSIONS

1. The asymmetry coefficient increases from $a = -0.09 \pm 0.01$ in zero field to $a = -0.29 \pm 0.01$ in fields of 17 — 27 koe.

2. The measured field dependence of the asymmetry coefficient in the range 0 — 17 koe does not agree with the hypothesis that the asymmetry depends linearly on $x^2/(1 + x^2)$, which would follow from (1) for the polarization when the Paschen-Back effect occurs in muonium.

3. The data are in good agreement with the hypothesis that μ^+ depolarization occurs through muonium formation if we assume either that several electron exchanges occur after stopping or that a few charge exchanges occur toward the end of the slowing-down process. Our data do not permit us to make a distinction between the two mechanisms for additional "dilution" of the initial polarization. Electronic measurements[9] indicate that the time during which the depolarization occurs is considerably shorter than 10^{-6} sec. This suggests that (7) is to be favored over (8), although both formulas agree equally well with our data. The charge-exchange mechanism[10] also insures very rapid depolarization and its great advantage over the electron-exchange mechanism lies in its universality — charge exchange of muonium may occur in all substances, whereas electron exchange is far from being everywhere possible.

4. The observed asymmetry coefficient $a = -0.29 \pm 0.01$ in the maximum magnetic fields is 10% smaller than the value $a = -1/3$ predicted by the V-A theory of weak interactions.[12] This discrepancy is larger than the experimental error and cannot be accounted for by systematic errors. Its cause lies either in additional 10% depolarization in an emulsion that is not restored by a magnetic field (the mechanism of which is not known) or in the failure of the V-A interaction to account for asymmetry in $\mu \rightarrow e$ decay.

5. In the course of the present and earlier work[2] we have examined about 340,000 $\pi^+ \rightarrow \mu^+ \rightarrow e^+$ decays, and have hunted especially for three-electron μ^+ decays according to the schemes $\mu^+ \rightarrow 3e$ or $\mu \rightarrow e + \bar{\nu} + \nu + \gamma$ with internal conversion of a gamma ray into a pair (a few decays of this second type were observed in references 13 and 14). Although these decays should be observable with efficiency close to unity we did not find a single reliable instance, while a few apparent cases of the second type can be accounted for by chance superposition of a slow-electron track.

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COSMIC RAYS FROM THE SUN

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A number of cosmic-ray flares were observed in the stratosphere at 64° geomagnetic latitude. The flares were due to 100-200 Mev protons, and were preceded by solar chromospheric flares which provided the source of the protons. It is suggested that the protons are carried by solar corpuscular beams with frozen-in magnetic fields.

LARGE increases in cosmic-ray intensity in the stratosphere were recorded at 64° geomagnetic latitude during July 9-21, 1959. These cosmic-ray flares were preceded by solar chromospheric flares with intensity 3^+ (on July 8, 10, 14, and 16, 1959), similarly to the events which had been observed previously, on July 8, 1958 and May 11-15, 1959.[1,2]

The measurments were carried out by means of cosmic-ray radiosondes[3] which rose to the stratosphere. The number of counts in a single Geiger-Müller counter and of double coincidences in a telescope incorporating a 7 mm aluminum absorber was recorded.

Twenty measurements were carried out during the above period, ten of which were made between July 9-14, and ten between July 15-21. Data of the single counter are shown in Fig. 1. The experimental points correspond to data averaged over 2-3 minutes of measurement.

It can be seen from Fig. 1b that, on July 15 and 17, an increase in cosmic-ray intensity was observed even at relatively great depths: on July 15 down to 160-170 g/cm^2, and on July 17 down to 300 g/cm^2. The number of particles increased very rapidly with decreasing pressure. At low pressures, the normal operation of the apparatus was disrupted because of the large counting rate ($> 4 \times 10^4$ min^{-1}). Fully reliable data could, however, be obtained as long as the counting rate was not greater than 1.5×10^4 min^{-1}.

Two measurments with a ~ 2.5 hr interval were carried out on July 11, and two measurements with a 4.5 hr interval were carried out on July 12. It can be seen from Fig. 1a that the results of two measurements on the same day are practically identical. It follows that the amplitude of the cosmic-ray flare remained constant for at least several hours, or changed only little, as happened, e.g., on July 15 and 17 (see Fig. 1b). Taking also into account that the measurements at high altitudes in the stratosphere lasted for about 30-40 min, one can assume that the results of the altitude-variation measurements were obtained for a practically constant flare amplitude. This makes it possible to obtain definite information concerning the nature and spectrum of the primary particles by studying the absorption curve of the additional cosmic rays in the stratosphere.

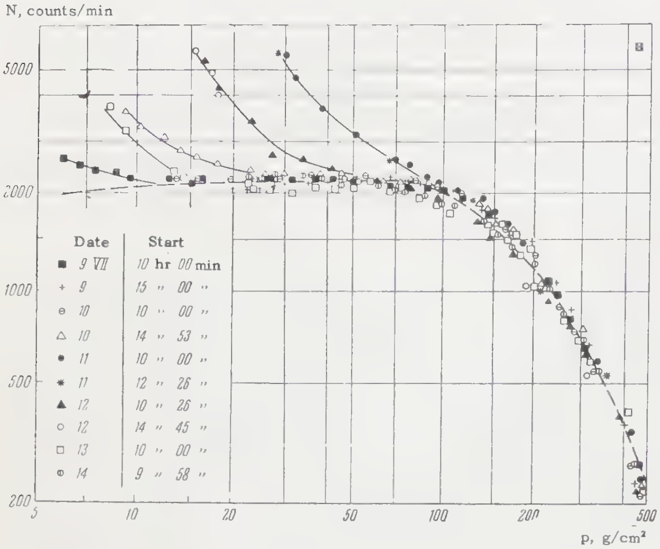
Let us compare the data given in Figs. 1a and 1b. It is clear that the results of the measurements for July 9, 10, and 11 at > 100 g/cm^2 pressure lie somewhat above, or at least are close to the normal intensity curve (dashed line), while the data obtained on July 12 and later lie below that curve. The latter fact is due to the effect of the decrease in main flux of the primary cosmic radiation during magnetic storms which occured after July 11.

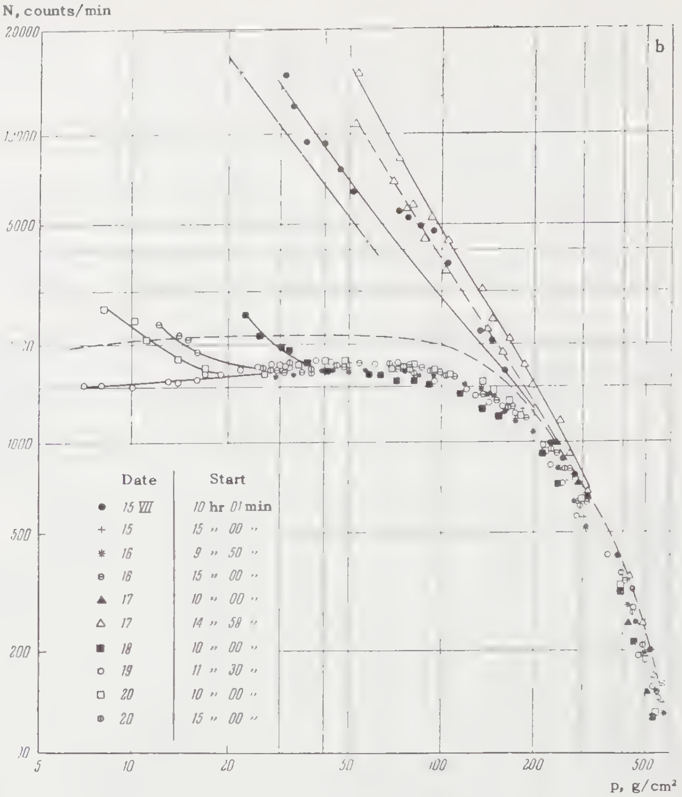
No increase in cosmic-ray intensity was detected during simultaneous measurements in the stratosphere at geomagnetic latitudes of 51° and 41°, where the critical energies for the primary particles are considerably greater than at 64° latitude. On the contrary, a decrease in cosmic-ray intensity was observed at 51° and 41° latitude at the time of the increase in cosmic-ray intensity at 64° accompanied by magnetic storms.

According to the data of July 18, the intensity decrease at 51° latitude reached 26%. This was the greatest decrease observed in the course of our measurements during the IGY.

It should be noted that this maximum cosmic-ray intensity decrease corresponds with the maximum amplitude of the flare at 64°. For the flare of July 15, which had a smaller amplitude, the intensity decrease at 51° and 41° latitude was also somewhat smaller. The decrease in cosmic-ray

FIG. 1. Variation of the counting rate N of the single counter with atmospheric pressure during: a) July 9–14, 1959; b) July 15–20, 1959. (dashed curves represent the normal intensity).





intensity in the stratosphere at 51° and 41° latitude was more limited during the flare of July 8, 1958, whose amplitude was still smaller. The small flares at 64° latitude (July 26 and October 3, 1958 and July 9, 1959) were not accompanied by any noticeable decrease in cosmic-ray intensity at 51° and 41° latitude. It is not possible at present to establish an exact correspondence between the amplitudes of cosmic-ray flares in the stratosphere and the amplitudes of the intensity decreases observed at 51° and 41° latitude. It follows, however, from our data that the two phenomena are interrelated.

Data on the additional number of particles, i.e., the difference between the measured and normal cosmic-ray intensity, are given in Fig. 2. Data for July 8, 1958 and May 11, 1959, are also included in the figure. It can be seen that the measurements taken during the various flares, at different times and at different altitudes, fall rather regularly along straight lines with a practically identical slope. An exception, in this respect, is the data of July 15, when a deviation at pressures in the range 70 — 100 g/cm^2 was observed during a short period of measurements (~ 7 min); later, the experimental points fell again on the straight line. Hence, one can conclude that the nature of the primary flare particles is identical not only during the course of each flare, but also in all flares. Such a conclusion, which in itself is important, makes it also possible to compare rather accurately the amplitude of the primary cosmic-ray flares.

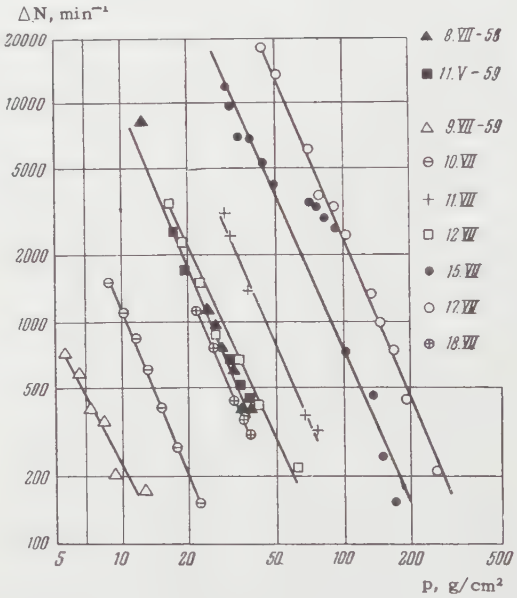


FIG. 2. Variation of the additional number of particles ΔN (single counter) with atmospheric pressure, according to the measurements of July 9–20, 1959.

During the measurements of May 12, 1959, when the telescope apparatus attained a great altitude (p = 6 g/cm^2), it was established that the number

Measurements during July 1959			Previous measurements		
Number	time	m	Date	time	m
9	8 hr 30 min	2.4	17.III—58	11 hr 18 min	35
10	13 » 30 »	7	8.VII	8 » 30 »	40
11	8 » 30 »	200	26.VIII	8 » 30 »	2.4
11	11 » 00 »	200	3.X	8 » 30 »	2.5
12	9 » 00 »	60	11.V — 59	8 » 30 »	40
12	13 » 15 »	60	11.V	11 » 30 »	40
13	8 » 30 »	6	12.V	8 » 30 »	40
15	8 » 30 »	800	12.V	13 » 30 »	35
15	13 » 30 »	350	13.V	8 » 30 »	3
16	13 » 30 »	7	14.V	13 » 30 »	5
17	8 » 30 »	2000	15.V	13 » 30 »	2.5
17	13 » 30 »	2800			
18	8 » 30 »	40			
20	8 » 30 »	24			

of particles in the flare was 40 times greater than normal. In estimating the amplitudes of other flares, we shall use this figure and also the corresponding data of the measurements of May 11 and 12 taken by means of the single counter. (The results of all these measurements are sufficiently close.) The values of m (the relative increase in the primary cosmic-ray flux as compared to the normal), according to the measurements of July 9 — 20, and also our unpublished earlier results, are presented in the table.

The results of the measurement of the number of double coincidences (above the normal rate) as a function of pressure are shown in Fig. 3. It can be seen that the slopes of the straight lines corresponding to the measurements of May 12, and July 12 and 15 are practically identical. The results of the measurements of July 11, 1959 are somewhat different: at small pressures, the slope is smaller; at higher pressures, however, the slope is identical with that obtained in other measurements. It is possible that this feature of the data of July 11 is not due to chance, since the measurements were taken in the absence of a magnetic storm and, consequently, of a Forbush decrease in cosmic-ray intensity on the earth. The other measurements were taken during magnetic storms.

Thus, the observed difference in the slope of the lines can be used as an indication that the spectrum of the primary protons varies, depending on whether the earth is immersed in solar corpuscular streams during the period of measurement. It is found, moreover, that the spectrum of primary protons is softer when this is the case. If similar results are obtained in the future, then this will constitute an additional confirmation of our earlier conclusion[2] that the source of the primary protons are the solar corpuscular streams themselves.

The proton nature of primary particles has already been discussed.[2] Assuming that the primary protons are isotropic at the top of the atmosphere (which follows from emulsion experiments[4]), and using the data of Fig. 3, it is possible to use the altitude dependence to find the energy spectrum of the primary protons, taking their ionization losses in the atmosphere into account. The integral proton spectra obtained in such a way are shown in Fig. 4. The slopes of the lines 1, 2, and 3 correspond to integral energy-spectrum exponents in the range 5.0 — 5.5. The slope of the upper part of the curve 4 corresponds to an exponent equal to 4.0.

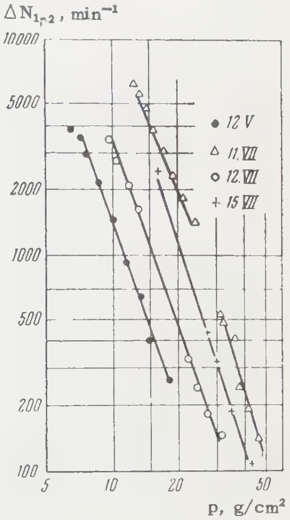


FIG. 3. Variation of the additional number of double coincidences with atmospheric pressure.

In the study of the flare of May 11 — 15, 1959,[2] the measurements of the telescope incorporating a 7 mm aluminum absorber and of the single counter were compared by a suitable recalculation. The comparison showed that the additional cosmic-ray intensity, according to the single-counter measurements, is roughly twice that calculated for a 25 g/cm^2 pressure. Consequently, in addition to the primary protons with range > 7 mm Al, particles with a shorter range are present in the stratosphere.

This result has also been confirmed by new measurements on July 9 — 21, 1959. It was found that the ratio of the rate measurements of the single counter and of the telescope increases with

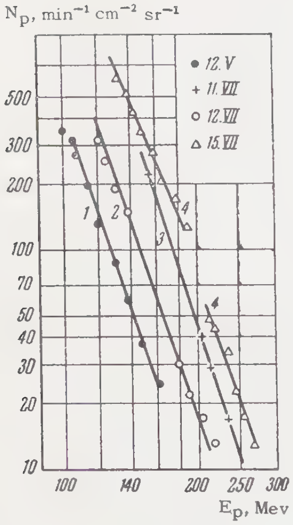


FIG. 4. Integral energy spectrum of primary protons.

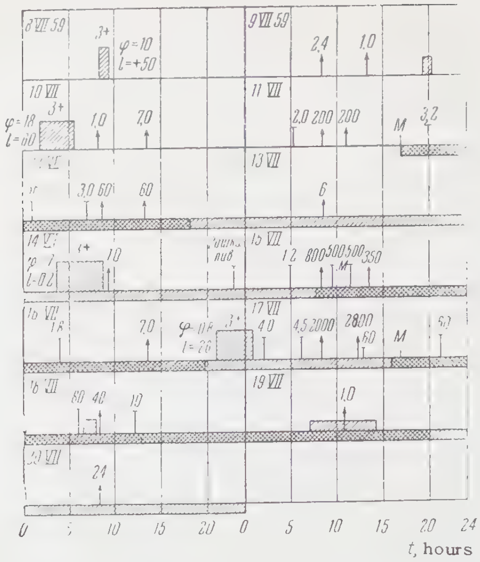


FIG. 5: Data on solar chromospheric flares, magnetic storms and cosmic rays (World Time). The arrows indicate the times of our measurements in the stratosphere, the signs ⊤ – those of Winckler.[5] The numbers above the signs give the ratio of the cosmic-ray intensity during the flare to the normal one.

the atmospheric depth of the observation level. Thus, e.g., the ratio of single-counter discharges to the number of double coincidences in the telescope, according to the measurements of July 15, equals 8 at a depth of 20 g/cm^2, and 30 at 60 g/cm^2.

The origin of the short-range particles (most probably electrons) which are detected by the single counter can be explained[2] by assuming that they are produced in the decay of evaporation neutrons from nuclear disintegrations produced by the primary protons. This hypothesis, however, has, so far, not been substantiated by calculation.

The Institute of Terrestrial Magnetism, Ionosphere, and Radio Wave Propagation of the Academy of Sciences, U.S.S.R. has supplied us with the data on chromospheric solar flares and magnetic storms during July 1959. Data referring to the period of July 8 – 20 are shown in Fig. 5. The height of the upper rectangles corresponds to the intensity of chromospheric flares, and the length of the base to their duration (φ and l are the heliographic coordinates of the chromospheric flares). The base of the lower doubly-shaded rectangles corresponds to the duration of magnetic storms and the Forbush decrease in cosmic-ray intensity. Single shading corresponds to the continuation of the Forbush decrease. The letter M indicates the onset of the Forbush decrease.

Disregarding the weak cosmic-ray flare on July 9, let us consider the data recorded after the large chromospheric flare of July 10. It can be seen from Fig. 5 that no increase in cosmic-ray intensity was detected in the measurement carried out on that day approximately 4 hr after the chromospheric flare. An intensity increase of m = 7.0 was found in the second measurement. It is now difficult to decide whether this increase was connected with the large chromospheric flare of July 10, or with the weak flare of July 9. (We believe the first case to be more probable.) One can maintain that the cosmic-ray flare recorded on July 11, 12, and 13 is connected with the large chromospheric flare of July 10, and that the cosmic-ray flare was delayed by more than 4 hr with respect to the chromospheric flare.

A great increase in cosmic-ray intensity on July 11 (m = 200) was recorded before the onset of the magnetic storm and the Forbush decrease which started at about 17:00 on July 11.

Winckler,[5] using an ionization chamber and a scintillation counter, also observed a great increase in cosmic-ray intensity in the stratosphere in Minneapolis (55°N) (see Fig. 5). For a correct comparison of our results with those of Winckler, it is necessary to remember that the critical energies for primary protons in Minneapolis and in Murmansk are 300 and 100 – 120 Mev respectively. In addition, the comparison should be made for a period during which the earth's magnetic field was not perturbed. Thus, the data for July 11 seem most suitable for a comparison.

It can be seen from Fig. 5 that, on that day, we have m = 200 for our two measurements, while, according to the data of Winckler, m = 2, i.e., a difference of about 200 in the additional flux is observed. One can easily see that the difference can be explained on the basis of the energy spectrum of primary protons with power exponent 5.0 and the values of the critical energy at the two latitudes. A comparison of our and Winckler's data obtained during magnetic storms will be less definite, since the additional flux of primary protons recorded in Minneapolis will vary very strongly with the perturbation of the earth's magnetic field because of the very steep spectrum of primary protons. This is the reason why the time variations of the flare amplitude, obtained at that latitude, do not, to a sufficient extent, reflect the true character of the time variations of the flare.

A strong chromospheric sun flare was again detected on July 14. Stratospheric measurements were carried out by us on that day towards the end of the flare. It was found that m = 1, which is natural if one takes the long delay of cosmic rays into account. However, the measurements on July 15 revealed a great increase in cosmic-ray intensity (m = 800); moreover, the amplitude of the flare decreased faster than in the previous case.

Another strong solar chromospheric flare was recorded on July 16. It was followed by a very strong (m = 2800) cosmic-ray flare. On July 18, the intensity decreased to m = 40. It can be seen from Fig. 5 that periods of magnetic storms with a sudden onset and Forbush decreases in the cosmic-ray intensity occured after the three large solar chromospheric flares which produced cosmic-ray flares in the stratosphere.

Thus, during the period of July 8 — 20, four chromospheric flares, each of several hours duration, caused cosmic-ray flares which continued, to various degrees, for 12 — 13 days.

The time variation of the cosmic-ray flare amplitudes is shown in Fig. 6. The time corresponding to the center of the chromospheric flare has been taken as zero-point reference. Curve 1 refers to the flare of July 10. For comparison, data for May 11 — 15, 1959 (curve 3) are also given, in addition to measurements at sea level of the cosmic-ray flare on February 23, 1956 (curve 2). It can be seen that considerable differences exist in the time variations for the stratospheric flares and the one detected at sea level.

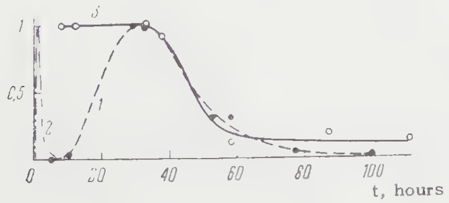


FIG. 6. Amplitude of cosmic-ray flares (arbitrary units) as a function of time. 1, 3 — stratospheric measurements, 2 — measurements at sea level on February 23, 1956.

The following conclusions may be drawn:

1. Great increases in cosmic-ray intensity (flares) observed in the stratosphere at northern latitudes are due to primary protons of solar origin. The energies of these protons, which follow from the measurements at 64° geomagnetic latitude, are greater than 100 — 120 Mev, and the differential energy-spectrum exponent equals 6.0. The energy spectrum of the primary protons differs little for the various flares.

2. All the five large cosmic-ray flares observed were preceded by solar chromospheric flares of maximum intensity (3^+).

3. Cosmic rays are delayed with respect to the chromospheric flares by more than 4 — 5 hours, but apparently less than 10 — 15 hours. Such long delays are not compatible with the velocities of primary protons.

4. The detected cosmic-ray flares are correlated with magnetic storms having a sudden onset, and with Forbush decreases of cosmic-ray intensity on the earth.

5. Magnetic storms have little influence on the primary-proton intensity in flares. A direct correlation has been established between the amplitudes of cosmic-ray intensity increases in the stratosphere at northern latitudes (64°) and the amplitudes of cosmic-ray intensity decreases in the stratosphere in more southern latitudes (51° and 41°).

6. The observed cosmic-ray flares have a duration of the order of days.

In order of explain satisfactorily all available data, it is necessary to assume that the solar corpuscular streams with frozen-in magnetic fields are themselves the source of the primary protons. The frozen-in magnetic fields constitute traps which can contain protons with the energies under consideration, produced or accelerated during solar chromospheric flares. Because of the fact that the protons in this energy range leave the corpuscular streams before these reach the zone of action of the earth's magnetic field, it is necessary to make the additional assumption about the existence of a magnetic medium in the interplanetary space.

In conclusion, the authors express their gratitude to Corresponding Member of the U.S.S.R. Academy of Sciences S. N. Vernov for his comments and his interest in the present article.

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ON AZIMUTHAL INSTABILITIES OF CIRCULATING CURRENTS

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Experimental data are presented from which it is deduced that circulating currents are unstable and break up into clusters which are uniformly spaced in the azimuthal plane. The conditions within these clusters are such that peculiar radial phase oscillations are set up. Because of this effect, collisions with the injector or wall of the vessel cause the electrons to be efficiently captured in betatron-type instruments.

A consideration of the azimuthal effects of space charge in a synchrotron, where longitudinal interaction of particles in clusters clearly occurs, has shown that, under certain conditions, these effects can noticeably influence the motion of the particles.[1] From the report of Kolomenskiĭ and Lebedev[2] it follows that these effects must also be taken into account in other systems, because of the instability of the azimuthally uniform distribution of the beam.

The results given below have been obtained by investigating electron capture in a betatron-type device of large cross-sectional area ($R_{\text{int}} = 10$ cm, $R_{\text{ext}} = 50$ cm, effective height 10 cm) at a constant magnet-gap field $H \sim r^{-n}$, $n \sim 0.5$. A ribbon-cathode injector was used[3] and the thickness of the injector blade was 4 mm. The injection voltage pulse ($U = 10$ to 50 kv) was rectangular in shape and 10^{-5} sec in duration: the repetition frequency was 50 cps. By selecting a value of the constant magnetic field for a given U, it was possible to vary the radius of the instantaneous orbit, $r_i = r_0 + a$, where r_0 is the injection radius.

Using probes to investigate the current circulating in the vessel at $a/r_0 \sim 0.1$, L. N. Bondarenko and A. A. Naumov found in 1955 that a pulse of high-frequency oscillations was produced in the probes at the instant of injection. However, neither the basic laws governing these hf oscillations nor the excitation mechanism were elucidated. In our experiments we have established the following facts:

1. Maximum intensity and stability of the hf signal are observed at values of r_i which roughly correspond to the maximum of the circulating current ($a/r_0 \sim 0.1$).

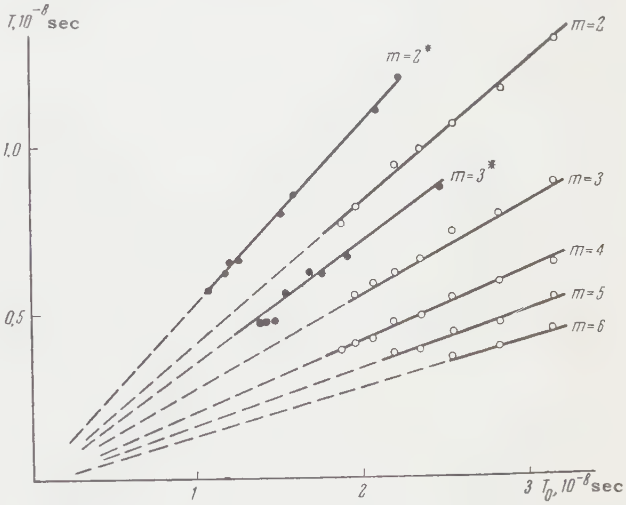
2. Hf oscillations are observed over the whole range of the variables covered by the experiments, i.e., $U = 10$ to 50 kv and injection current $J = 5$ to 500 ma. At $U = 20$ kv, hf oscillations occur whenever $J > 2$ to 5 ma.

3. The frequency of the oscillations is independent of J, of the pressure P in the vessel ($P = 5 \times 10^{-5}$ to 3×10^{-6} mm Hg), and of the position of the probes.

4. The amplitude of the hf signal is a maximum if the probe lies near the beam. The maximum amplitude is of the order of hundreds of volts.

5. The period of the hf oscillations is $T = \kappa T_0/m$, where $m = 2, 3, 4, \ldots$ and $T_0 = 2\pi r_i/v$: $\kappa \approx 1.1$ when $r_0 = 20$ cm and $a > 0$, and $\kappa \sim 0.85$ when $r_0 = 40$ cm and $a < 0$ (see diagram). We observe that κ differs from unity because the mean energy of the particles that form the clusters differs from the mean injection energy (see below).

From the results enumerated above, it follows that part of the circulating current breaks up into electron clusters of equal density. These clusters are uniformly distributed in the azimuthal plane



Relationship between the period T of the hf oscillations and the period of revolution of the particles T_0 ($a/r_0 \approx 0.1$). The lines marked with an asterisk are obtained with $r_0 = 20$ cm and $a > 0$, and the remaining lines are obtained with $r_0 = 40$ cm and $a < 0$.

(the latter fact is a consequence of the absence of the first harmonic $T = T_0$). Actually, since a uniform azimuthal particle distribution is unstable,[2] the electrons must group themselves in condensed clusters. Since the neighboring particles in the azimuthal plane are attracted to the cluster, this process will continue until the force acting on them changes sign owing to the effects of particles left behind. It is clear that these particles can form one more or several more self-sustaining clusters, which indeed explains the absence of the first harmonic $T = T_0$.

The occurrence of azimuthal non-uniformities of the circulating current can have an important effect on the capture efficiency in betatrons. Since, for weakly-focused systems, $d\omega/dW < 0$ (ω is the rotation frequency and W the energy of the particles), and since the azimuthal component E_θ of the field of a cluster changes sign at the center of the cluster, it is evident that conditions exist in the cluster for the occurrence of unique radial-phase oscillations of the particles. If the injection current is sufficiently large, noticeable condensation of clusters occurs even after a few revolutions. Also, the constituent particles "separate" according to their radial position and energy [when $r_i = 20$ cm, $U = 20$ kv, $n = 0.5$, and $E_\theta = 1$ to 5 v/cm, the change in r_i (for electrons) is 0.6 — 3 mm within 5 revolutions]. The first particles to be lost by collision with the injector or the walls of the vessel are those whose instantaneous orbits lie nearest to the injector or walls. As a result of the loss of these particles, the mean energy and mean orbit radius of the remaining part of a cluster increase (in the case of internal injection) or decrease (external injection), and some part of the cluster is trapped after a number of collisions. Under conditions of continuous injection, the injected electrons should group themselves near the clusters formed in the initial instants of injection, i.e., the clusters must condense until a dynamic equilibrium is established between the particles that are being injected and those lost. This argument is supported by the constancy of the frequency and amplitude of the hf oscillations over the flat part of the injection pulse.

It is an interesting fact that the circulating current (recorded by a Rogovski coil) and the hf oscillations of the same frequency are observed long after the trailing edge of the injection pulse (up to hundreds of microseconds at $P = 5 \times 10^{-6}$ mm Hg). The strengths of these oscillations and of the circulating current are 10 to 30% of the corresponding values at the instant of injection, and decrease more rapidly with time with increasing pressure. (Currents occurring after the injection pulse are discussed also by Ivanov, Komar, and Korobochko.[4])

In concluding we note that the capture mechanism proposed above, based on a longitudinal interaction of the electrons, may prove more effective than the mechanism associated with transverse interaction of the particles in the beam.[5,6] Information on the relative efficiencies of these mechanisms can be obtained experimentally by using an injector mounted above the plane of the equilibrium orbit.[6]

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SOVIET PHYSICS JETP VOLUME 12, NUMBER 2 FEBRUARY, 1960

SECOND-ORDER QUADRUPOLE EFFECTS IN SODIUM NITRATE

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J. Exptl. Theoret. Phys. (U.S.S.R.) **39**, 260-261 (August, 1960).

A shift of the central nuclear magnetic resonance line of Na^{23}, due to second-order effects, was observed in sodium nitrate single crystals. The sign and the magnitude of the shift agree with theoretical values.

NUCLEAR magnetic resonance of Na^{23} in sodium nitrate crystals was first investigated by Pound.[1] Pound observed a spectrum consisting of three lines, a central line and two symmetrically situated satellites. Such a spectrum is due to the quadrupole interaction of the Na^{23} nuclei with the electric field gradient in the crystalline lattice.

Assuming the quadrupole interaction energy of the nuclei to be sufficiently small compared with their energy of interaction with the external magnetic field, and therefore employing first-order perturbation theory, Pound showed that the magnitude of the splitting is given by the following formula

$$\Delta\nu = \frac{3(2m-1)}{8I(2I-1)}\frac{e^2qQ}{h}(3\cos^2\varphi - 1), \tag{1}$$

where I is the nuclear spin (equal to $^3/_2$ in the case of Na^{23}), Q is its quadrupole moment, m is the magnetic quantum number, eq is the component of the gradient along the symmetry axis, and φ is the angle between the symmetry axis and the magnetic field.

According to Pound's experimental data the coefficient e^2qQ/h is equal to 334 kc/sec.

As may be seen from (1), in the first approximation a shift of the satellite frequencies occurs (the $m \rightarrow m-1$ transitions for $m \neq {}^1/_2$), while the central line ($m = {}^1/_2 \rightarrow m = -{}^1/_2$) remains undisplaced. However, as will be shown below, the energy of the quadrupole interaction of the Na^{23} nuclei in sodium nitrate is not sufficiently small to enable us to limit ourselves to only the first approximation of perturbation theory, i.e., to consider that there is no shift of the central line.

Second-order perturbation theory yields (cf., for example, reference 2) a shift of the central line which in the case of the Na^{23} nuclei is given by

$$\Delta\nu_c = \frac{3}{64\nu_0}\left(\frac{e^2qQ}{h}\right)^2(1 - 9\cos^2\varphi)(1 - \cos^2\varphi). \tag{2}$$

The shift increases as the operating frequency ν_0 is diminished. For the frequency $\nu_0 = 5.2$ Mc/sec which corresponds to a field of 4600 oe, and for the value of e^2qQ/h quoted previously, the shift lies in the range from +1.0 to −1.8 kc/sec. Such a shift of the central line can be easily observed experimentally.

In order to obtain this shift experimentally, we have recorded by means of the apparatus described previously[3] nuclear resonance signals from Na^{23} in a single crystal of $NaNO_3$ and in a saturated aqueous solution of NaCl. The absorption line of Na^{23} in solution was used as the reference point with respect to which the position of the absorption line of Na^{23} in sodium nitrate was measured.

Experiment showed that in accordance with (2), the two lines coincide for $\varphi = 0°$, i.e., $\Delta\nu_c = 0$, while the biggest shifts, which differ in sign, occur for $\varphi = 42$ and 90°. Within the limits of experimental error, the magnitudes of the shifts correspond to the theoretical expression (2).

The dependence of the shift of the central line on the orientation of the single crystal must lead to a broadening of this line in a powdered sample, since such samples contain small crystals of random orientation. Indeed, experiment shows that in powdered sodium nitrate the absorption line is broadened by a factor of approximately 1.5 while retaining its full intensity.

Moreover, it can be seen from (1) and (2) that for no orientation of the single crystal is it possible to obtain complete coincidence of the satellites and the central line, and this is also confirmed by experiment.

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NUCLEAR MAGNETIC RESONANCE IN PLASTICALLY DEFORMED ROCK SALT

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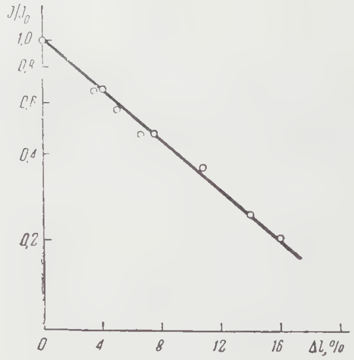
The intensity of the nuclear magnetic resonance line of Na^{23} was measured in plastically deformed rock salt crystals as a function of the degree of deformation.

THE magnetic resonance of nuclei possessing quadrupole moments is very sensitive to different kinds of crystal defects and, in particular, to defects due to plastic deformation. The effect of plastic deformation on the resonance of Br and I nuclei in KBr and KI crystals has been studied in a number of papers.[1,2] However, the quadrupole interaction of these nuclei is so large that even in the original undeformed crystals only the central line is observed, while the satellites are completely smeared out, owing to the presence of defects arising during the process of crystal growth. Therefore, in the case of KBr and KI crystals, we are unable to observe the behavior of the satellites under plastic deformation. At the same time, an investigation of the effect of plastic deformation on the satellites is of considerable interest, particularly because a number of papers[3,4] report analogous investigations carried out on crystals with impurities. Apparently it is possible to observe the behavior of the satellites under plastic deformation only in a crystal in which either the concentration of the growth defects is smaller than in KBr and KI crystals, or the quadrupole interaction of the nuclei whose resonance is being investigated is less than the quadrupole interaction for the Br and I nuclei. A suitable crystal, from the last mentioned point of view, is rock salt, NaCl. Experiments on the measurement of the intensity of the absorption line due to Na^{23} in NaCl crystals, carried out by Kawamura et al.,[3] have shown that in these crystals both the central line and the satellites can be fully observed. Thus, in the case of rock salt there exists a possibility of studying the effect of plastic deformation on the satellites.

The measurements to be described have been carried out utilizing an apparatus of Pound's type.[5] The rock salt crystals were deformed by a linear compression along the [001] direction. As a rule, only one pair of mutually perpendicular glide systems took part in the deformation, so that the deformed samples had the shape of a double-ended barrel. For each sample the measurements were carried out for three different orientations of the magnetic field: parallel to the [100], [110] and [010] directions. No angular dependence was observed in the course of these measurements.

The diagram shows on a semilogarithmic scale the relative intensity of the satellites (i.e., the total intensity minus the intensity of the central line) as a function of the degree of deformation. Each point represents an average of measurements corresponding to the three orientations mentioned previously. Deviations from these average values do not exceed 6%.



Dependence of the relative intensity of the satellites on the degree of deformation.

As can be seen from the diagram a decrease in the intensity of the satellites occurs as a result of plastic deformation. It must be emphasized that the width of the "total" line remains, within experimental error, the same as for undeformed samples.

Defects arising in the course of plastic deformation are linear ones, with a definite orientation in the crystal (dislocations). However, the absence of any angular dependence compels us to regard them in our case either as approximately equiaxial (short segments of dislocations), or as randomly oriented ones. The former case appears to us to be more probable, and therefore we shall speak henceforth not of dislocations but of some centers of distortion.

Taking this into account, and also the absence of any broadening of the absorption line, we can, just as in the case of crystals with impurities, utilize the critical-sphere model.[3,4] According to this model, the intensity of the satellites in the case of a random distribution of centers of distortion must be proportional to the probability of the center of distortion lying outside the critical sphere, i.e.,

$$J/J_0 = (1-c)^{v_c/v_0}, \tag{1}$$

where J and J_0 are the intensities of the satellites in deformed and undeformed crystals respectively, c is the concentration of centers of distortion, v_0 is the volume occupied by a center of distortion, and v_c is the volume of the critical sphere. On taking the logarithm of (1), and on taking into account the fact that c is small we have

$$\ln(J/J_0) \approx -cv_c/v_0. \tag{2}$$

It appears natural to assume that the concentration of the centers of distortion is proportional to the degree of deformation. Then, a plot of $\ln(J/J_0)$ vs. the degree of deformation should be a straight line, and this is indeed the case as can be seen in the diagram. Since c and v_0 are not known we can not determine v_c from our experimental data. Nevertheless, we have the possibility of making a rough estimate of the dimensions of the critical sphere by utilizing the data of Kawamura et al.,[3] who determined, on the basis of their experimental results, the radius of the critical sphere for Na^{23} nuclei in mixed NaCl-NaBr crystals.

Following Bloembergen,[6] we can write the expression for the electric field gradient in the lattice in the cases of a mixed and of a plastically deformed crystal. In a mixed NaCl-NaBr crystal the gradient at a distance r from an ion of the Br^- impurity is given by

$$eq = 6e(a_{\text{imp}} - a_0)a_0^2/a^3r^3, \tag{3}$$

where a_{imp} and a_0 are the radii of the Br^- impurity and of the principal Cl^- ion, and a is the lattice constant of NaCl. In plastically deformed NaCl, the gradient at a distance r from a dislocation is given by

$$eq = 6e/a^2r. \tag{4}$$

At the surface of the critical sphere the two gradients must be equal. On equating (3) and (4), and on taking into account the fact that in the former case the radius of the critical sphere is approximately equal to 6 A, we find that in plastically deformed NaCl the radius of the critical sphere is of the order of 10^3 A. Of course this figure requires experimental verification.

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SOVIET PHYSICS JETP VOLUME 12, NUMBER 2 FEBRUARY, 1961

PARAMAGNETIC RESONANCE IN A QUASI SINGLE CRYSTAL OF $CrCl_3$

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Paramagnetic resonance in a quasi-single-crystal of $CrCl_3$ has been measured at room temperature at a frequency of 36,000 Mc/sec. An anisotropy of the g factor has been observed ($g_\perp = 1.989 \pm 0.001$; $g_\parallel = 1.984 \pm 0.001$) which is accompanied by a change in the width of the absorption line $\Delta H_\perp = (98 \pm 2)$ oe; $\Delta H_\parallel = (140 \pm 5)$ oe.

ANHYDROUS chromic chloride has several times been studied by means of the electron paramagnetic-resonance (e.p.r.) method (cf., for example, references 1 and 2). As far as we know, however, polycrystalline specimens were used in all cases. This can evidently be explained by the fact that it is impossible to grow sufficiently large single crystals of this compound.

Chromic chloride has a hexagonal layer-like lattice in which the chromium atoms form layers bounded on both sides by layers of chlorine atoms.[3] $CrCl_3$ crystallizes in the form of small flakes whose surfaces are parallel to the chromium and chlorine layers. It is, therefore, natural to assume that a $CrCl_3$ specimen consisting of a number of flakes laid on each other possesses the main properties of an axially symmetric single crystal.

The e.p.r. of such quasi-single-crystal specimens of $CrCl_3$, consisting of three to five flakes, was studied at 36,000 Mc/sec at room temperature by a previously described method.[4] As a result of the measurements we obtained the following values of the g factor and widths of the e.p.r. lines:

$$g_\perp = 1.989 \pm 0.001, \qquad g_\parallel = 1.984 \pm 0.001,$$

$$\Delta g = 0.005 \pm 0.002, \qquad \Delta H_\perp = (98 \pm 2)\ \text{oe},$$

$$\Delta H_\parallel = (140 \pm 5)\ \text{oe}.$$

Thus, it can be seen that in the quasi single crystal of $CrCl_3$ a marked anisotropy of the g factor is observed which is accompanied by a change in the width of the resonance curve (the width of the curve was measured at the half-height of the intensity).

One can speak of several possible reasons leading to a change in the width of the e.p.r. line in the directions parallel and perpendicular to the symmetry axis of the crystal. It is known, for instance, that although fine splitting in undiluted salts is averaged out by exchange interactions, the fine structure does contribute to some extent to the broadening of the e.p.r. line.[6] It would, however, be incorrect to assume that the difference of this contribution for different directions determines the observed line widths in $CrCl_3$, since in experiments conducted at frequencies differing by more than a factor of 3500 the widths of the e.p.r. lines are approximately the same: at 10 Mc/sec (reference 1), and 9308 Mc/sec (reference 2) in polycrystals ΔH is 132 and 148 oe respectively. The widths obtained from our measurements at 36,000 Mc/sec do not exceed these values.

The observed widths of the e.p.r. lines also cannot be explained by a difference in the dipole-dipole interactions in the investigated directions. Actually, the widths calculated from Van Vleck's formula,[5] assuming that the line has a Gaussian shape, are 900 oe in a constant magnetic field parallel to the symmetry axis of the crystal, and 1030 oe for the two directions of H_0 perpendicular to this axis. Such a large discrepancy in the observed and calculated widths indicates the presence of strong exchange interactions between the Cr^{3+} ions which has already been previously noted.[2]

On the other hand, a comparison of the calculated and measured values of ΔH allows us to assume that the directional dependence of the e.p.r. line width in $CrCl_3$ is due to the anisotropy of exchange interactions. The same conclusion could have been reached from a consideration of the structure of $CrCl_3$: neighboring Cr^{3+} ions belonging to the same layer are at a distance of 3.5 A from each other, while the distances between nearest Cr^{3+} ions of neighboring layers are 6.1 A. However, there remains here the unestimated role of the chlorine ions through which a superexchange may occur.[7] Employing the approximate formula, proposed by Anderson and Weiss[6]

$$\Delta\omega = [{}^{10}/{}_{3}\,\Delta\omega_p^2]/\omega_e,$$

where ω and ω_p are the observed and dipole-dipole half-widths of the e.p.r. lines at the half-height of the intensity, it is possible to estimate the frequencies ω_e of the exchange interactions: the exchange frequency in the direction perpendicular to the symmetry axis of the crystal $\omega_{e\perp} = 3.2 \times 10^{11}$ cps, while $\omega_{e\parallel} = 1.7 \times 10^{11}$ cps.

The above method of investigation can also be applied to other substances with a layered structure.

In conclusion, the authors express their gratitude to Prof. B. M. Kozyrev who suggested the topic of this work.

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SOVIET PHYSICS JETP VOLUME 12, NUMBER 2 FEBRUARY, 1961

TRANSVERSE MOMENTUM COMPONENT OF STRANGE PARTICLES PRODUCED IN PENETRATING COSMIC RAY SHOWERS

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Results of measurement of the transverse momenta of strange particles produced by cosmic radiation are presented. It is shown that the mean value of the transverse momentum of heavy unstable particles is (0.52 ± 0.06) Bev/c. Within the limits of experimental errors it is independent of the type of strange particle and energy of the primary particle in the interval $(1-10)\times 10^{10}$ ev.

ONE of the most reliably established and interesting properties of nuclear interactions at high and very high energies is the constancy of the transverse momentum components of secondary π mesons and their comparatively narrow distribution near a mean value of 0.3 to 0.5 Bev/c. On the other hand, the transverse momentum of nucleons, antinucleons, and strange particles remains almost entirely uninvestigated. Perkins[1] and Takibaev[1] assumed, on the basis of indirect data, that P_t, the transverse momentum component of particles other than π mesons, has a broad distribution extending up to several Bev/c.

In the present work an attempt was made to measure directly the transverse momenta of strange particles (Λ^0, θ^0, and $\Sigma^{\pm}$), recorded in the Elbrus laboratory by an arrangement consisting of a double cloud chamber in a magnetic field, the cloud chamber being triggered by penetrating cosmic-ray showers. A description of the arrangement and the method of treating the data on the decay of unstable particles has been given in papers by the staff of the laboratory.[2-7]

Used in the study of the distribution of the values of P_t were twenty-five Λ^0, twenty-one θ^0, and thirteen $\Sigma^{\pm}$ particles for which the point of generation and the geometric parameters of the shower responsible for their production could be established.

For all the showers generated in the upper absorber, and for part of the showers generated by neutral particles in the absorber between the two working volumes of the chamber, the direction of the generating particle was taken along the shower symmetry axis that divided the number of shower particles in half. The error committed in determining the angles between the primary particle and the products of its nuclear interaction by this method was estimated by means of those cases for which the direction of the primary particle could be determined directly from its track in the upper volume of the chamber. To do this, we measured the angles of the shower tracks with respect to the direction of the primary particle and the axis of symmetry of the nuclear interaction. The mean-square error of the measurement of the angle between the primary particle and its products, determined from the difference between the respective angles, was 9°. It can be shown that such an error does not lead to any significant distortion in the shape of the distribution and in the mean value of P_t in the case of strange particles produced in interactions of energy of the order of several tens of Bev.

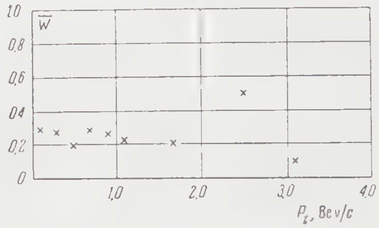

FIG. 1. Dependence of the experimentally observed mean probability of the recording of a decay $\overline{W}$ on the transverse momentum component P_t. The spread in the points at large P_t is due to the small number of cases. $\overline{W} = n^{-1}(\Sigma 1/W_i)^{-1}$.

Since the probability of recording the decay of an unstable particle in the chamber may depend significantly on the value of its transverse momentum, the distribution of the values of P_t should be corrected for the probability of recording each case. To do this, the distribution was constructed with each decay assigned a weight W_i^{-1}, where W_i is the a priori probability of its being recorded,

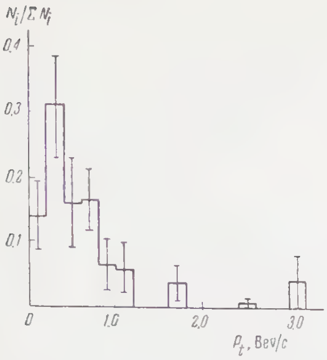

FIG. 2. Distribution of the transverse momentum components P_t for all strange particles.

and the error in the number of cases in a given interval of values of P_t was defined as $(\Sigma W_i^{-2})^{1/2}$. To demonstate the dependence of the probability of being recorded on P_t, Fig. 1 shows the probabilities W_i averaged over cases with close values of P_t. It is seen that, up to transverse momenta of the order of 2 Bev/c, the probability of recording a decay cannot distort the distribution.

	Number of cases	$\overline{P}_t$, Bev/c	$\overline{P}_t$ (Bev/c) omitting two cases with $P_t > 2$ Bev/c
All cases	59	0,65±0.07	0.52±0.06
Showers generated in upper absorber	29	0,69±0,15	0,47±0.10
Showers generated in absorber between chamber volumes	30	0.57±0.06	
Λ^0 hyperons	25	0.45±0.06	
θ^0 mesons	21	0.86±0.15	0,54±0.11
$\Sigma^\pm$ hyperons	13	0,72±0.19	0,64±0,14

Figure 2 shows the distribution of the values of P_t with corrections for the probability of recording the decay for all 59 cases of decay of strange particles included in this analysis. The table lists the corresponding mean values of the transverse momenta P_t. The spectrum of the value of P_t has the most probable value in the momentum region 0.2 to 0.4 Bev/c and extends up to 3 Bev/c. However, since the maximum measurable momentum in our arrangement was 2 Bev/c and the error in the sine of the angle between the line of flight of the unstable particle and the trajectory of the motion of the generating particle was of the order 0.12, both cases in which the transverse momentum is greater than 2 Bev/c may be due to errors in measurement. Since these cases, owing to their greater weight, produce a significant change in the mean values of the P_t distribution, the table also shows the means values without these cases taken into account. The mean square value of all transverse momenta is shown in the first row of the table.

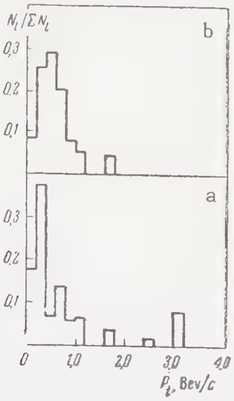

FIG. 3. Distribution of the values P_t: a – for particles generated in the upper absorber; b – for particles generated in the absorber between the two working volumes of the chamber.

Owing to the device for selecting the recorded interactions, the energy of the shower generated in the absorber above the chamber is several times the energy of the interactions generated in the absorber placed in the gap between the two volumes of the chamber. Therefore the separation of the overall distribution into cases recorded in the upper and lower volumes can give an idea of the energy dependence of the values of P_t of heavy unstable particles in the energy region $(1-10) \times 10^{10}$ ev. Shown in Figs. 3a and 3b are respective distributions, which, apparently, indicate the absence of a dependence of P_t on the energy, within the limits of experimental error. The weighted mean values of the transverse momenta are shown in rows 2 and 3 of the table.

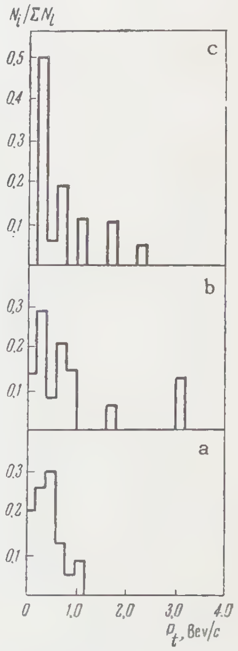

FIG. 4. Distribution of the values of P_t: a – for Λ^0 hyperons; b – for θ^0 mesons; c – for $\Sigma^\pm$ hyperons.

In Figs. 4a, 4b, and 4c the distributions of P_t are separated according to the type of unstable particle. These distributions do not differ from one another within the limits of statistical error. The corresponding weighted mean values of the transverse momenta are given in the last rows of the table.

In conclusion, the authors take this opportunity to express their gratitude to Z. Sh. Mandzhavidze and G. E. Chikovani for their interest in this work and for valuable comments, and also D. V. Gersamiya, A. A. Kozlov, D. M. Kotlyarevskiĭ, N. G. Tatalashvili, and A. I. Tsintsabadze for taking part in the operation of the equipment and analysis of the data.

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SOVIET PHYSICS JETP VOLUME 12, NUMBER 2 FEBRUARY, 1961

OBSERVATION OF μ-MESON BURSTS IN A SET OF IONIZATION CHAMBERS

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Bursts due to high-energy ($E \gtrsim 10^{11}$ ev) μ mesons moving at a small angle to the horizontal plane were observed with the aid of a set of cubic ionization chambers. It is shown that with a set of ionization chambers similar to that described in the paper, but somewhat modified, it should be possible to study the angular distributions of high-energy μ mesons.

APPARATUS

THE apparatus was designed for the study of the cores of extensive atmospheric showers and contained two layers of ionization chambers (see references 1 and 2). The chambers were placed side by side and formed an area $2 \times 2\text{m}^2$. A vertical section through the apparatus is shown in Fig. 1.

Each layer consisted of 64 chambers of cubic shape. The chambers were connected to separate channels of amplification and pulse registration. The range of the registered pulses corresponded to the passage of from 5 to 4×10^4 relativistic particles through the chamber volume. The registration of the pulses took place whenever the total ionizing burst in all chambers exceed a value corresponding to 1000 relativistic particles.

Working together with the apparatus was a system of a large number of Geiger-Müller counters arranged in groups at various distances.

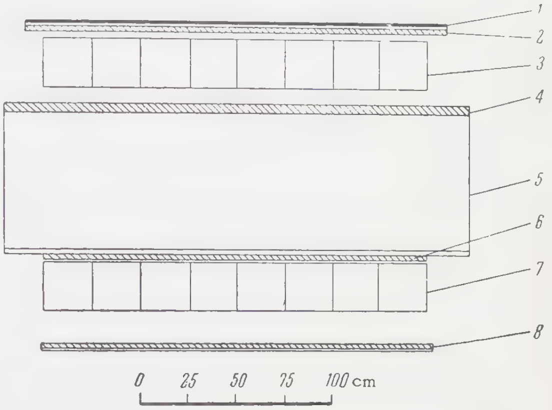


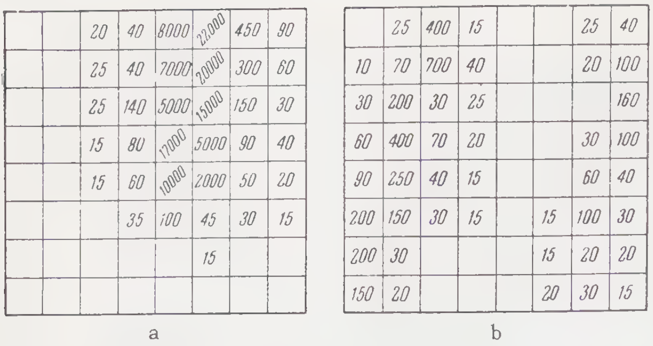
FIG. 1. Cross section of the apparatus: 1 – Al, thickness 0.3 cm; 2, 4, 6, 8 – Pb, thickness 2.5, 5, 2.5, 2 cm, respectively; 3, 7 – ionization chambers; 5 – graphite, thickness 73 cm.

RESULTS

A large part of the events registered by the apparatus was produced by air showers consisting both of electron—photon components and nuclear-active particles. The showers observed with the aid of the chambers were accompanied in most cases by the triggering of the counters located above the apparatus and around it. These cases have been analyzed previously.[1,2]

Among the registered events, however, there are cases similar to those shown in Fig. 2. The group of chambers with approximately the same size of bursts were located along a straight line forming an ionization distribution in the chambers that was "extended" in shape. The number of events of this type turned out to be the same for the upper and lower rows of the chamber. As a rule, such events are not accompanied by the triggering of the hodoscopic counters placed around the apparatus. The triggering of the counters (sometimes a few) was observed mainly when the shower intersected the upper row of ionization chambers. It is of interest that in this case the triggered counters were situated in a narrow strip in the direction of the extended distribution of the ionizing bursts. Evidently, the shower particles are scattered on the unshielded lead surface and enter the counters. The cases of the "extended" distribution of bursts in the chambers differ from the ordinary ones also in the fact that the passage of the particles is registered only in one of the layers during the total absence of bursts in the other, since the usual burst in one row of chambers, in a considerable part of the cases, is accompanied by bursts in the other row.

The character of the events do not allow them to be identified with air showers. Such events may be showers of relativistic particles generated in the lead over the chambers by single particles traveling at a small angle to the horizontal plane. Particles generating such showers cannot be nuclear-



a

		20	40	8000	22,000	450	90
		25	40	7000	20000	300	60
		25	140	5000	15000	150	30
		15	80	17000	5000	90	40
		15	60	10000	2000	50	20
			35	100	45	30	15
					15		

b

	25	400	15			25	40
10	70	700	40			20	100
30	200	30	25				160
60	400	70	20			30	100
90	250	40	15			60	40
200	150	30	15		15	100	30
200	30				15	20	20
150	20				20	30	15

FIG. 2. Examples of the distribution of ionization in the layer of ionization chambers in the cases of the registration of showers from oblique μ mesons; a represents the largest burst recorded by us; b is one of the two cases of registration of μ-meson double showers. The layout of a row of ionization chambers is shown; the numbers in the squares give the sizes of the burst in the given chamber (in numbers of relativistic particles).

active particles, since the intensity of the N component drops with an increase in the angle θ to the vertical as $\exp\{-\mu x(\sec\theta - 1)\}$, where for sea level $\mu x = 8$. For angles $\theta > 75°$ the intensity of the N component attains only the value $\sim 10^{-10}$ times the intensity along the vertical.

A quantitative analysis of the recorded events indicates that the observed bursts with an ionization distribution of "extended" shape may be explained as radiation showers from μ mesons. The analysis was made in the following way. We selected cases satisfying two conditions:

1. The ionization was concentrated only in one row of ionization chambers, and its total size corresponded to the passage of > 1000 relativistic particles.

2. There were four or more chambers, situated in a row along one line, in which ionization of comparable size was observed (no more than 1.5 times as great).

The number of relativistic particles in the shower was taken from the sum of the ionization produced in the chambers situated along a line perpendicular to the direction of propagation of the shower. The maximum sum from several measurements (along the shower) was taken. For each event the value of the total ionization was found in all chambers of a given row. It turned out that the number of cases selected in this way was 0.25 ± 0.03 per hour for each row of chambers (a total of 190 cases were recorded in both rows of chambers). In all cases, except one, the bursts in one row of chambers were not accompanied by bursts in the other row.

According to the data of Schein and Gill,[3] for a spherical chamber covered by 12 cm of lead, the number of μ-meson bursts caused by showers of 400 or more particles per cm^2 of chamber was $1.36 \times 10^{-6}\,\text{cm}^{-2}\,\text{min}^{-1}$.

The expected number of μ-meson showers in our arrangement can be roughly estimated, according to the data of Schein and Gill,[3] if it is assumed that the high-energy μ mesons have an isotropic distribution and that we recorded only μ meson bursts for which $\theta \geq 75°$ ($\tan\theta \gtrsim 4$), where θ is the angle between the trajectory of the meson and the vertical. It turned out that the experimental value was less than three times the calculated value. This should be regarded as satisfactory agreement if it is considered that there is a possibility of an appreciable increase in the number of particles in the shower as a result of the method we employed to determine it and also if one takes into account the large undefined effective area and effective solid angle for the cases we selected.

We note that, according to the estimate, the total number of μ-meson showers with a given number of particles passing through the apparatus is more than twenty times the measured value in the selected angular interval.

Shown in Fig. 3 is the integral spectrum of the observed showers obtained from the number of particles. Also shown there is the integral spectrum of these events under the condition that the number of particles in the shower was measured from the total ionization produced in all chambers.

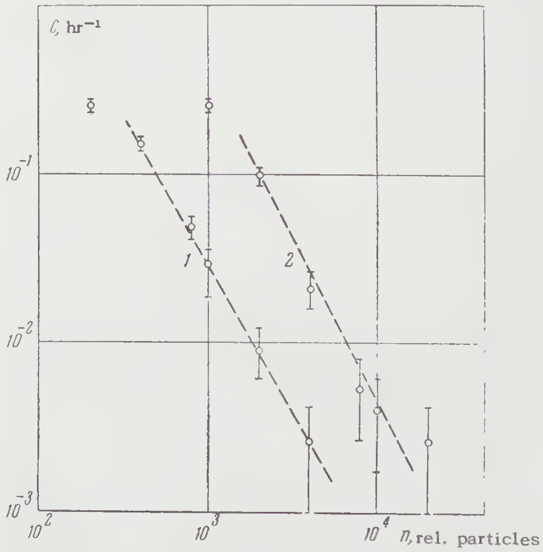


FIG. 3. Integral spectrum of the showers: 1 – from the number of relativistic particles in the shower; 2 – from the total ionization (in relativistic particles) in a row of ionization chambers. The number of showers recorded in one hour by one row of chambers is marked off on the axis of ordinates. The curves were constructed from the data obtained during 370 hours of operation of the apparatus.

The observed bursts should refer to μ mesons of energy of the order of 10^{11} ev and above. The results indicate that the number of bursts greater than a given size (n) decreases in accordance with a law of the n^{-2} type or somewhat more steeply.

It is seen from Fig. 3 that the number of showers of a given size measured from the total ionization produced by a shower in the entire layer of chambers is appreciably greater than the number of showers with the given number of particles. This results from the fact that a shower with n particles, in crossing the chambers at an angle θ to the vertical, produces in them a total burst corresponding to $n/\cos\theta$ particles, which, in the spectrum of bursts of the n^{-2} type, should cause the number of bursts to increase as $\cos^{-2}\theta$.

Curve 2 of Fig. 3 gives the number of bursts due only to μ mesons at an angle of inclination $\theta \geq 75°$. The total number of particles due to μ mesons from all directions should be several times greater. This should be borne in mind when an apparatus with large plane systems of chambers is used to determine the direction of the particle stream. At sea level the oblique μ-meson showers in such systems resemble the arrival of a "young" atmospheric shower[4] or nuclear-active particles of high energy.

Besides the described bursts in the form of single showers there were recorded, during a period of 370 hr of operation, two cases in which two, almost parallel showers were observed simultaneously. Since the probability for the formation of a radiation shower is very small ($\sim 10^{-3}$), the simultaneous observation of two showers suggests that the passage of two high-energy mesons at a small distance from one another is not a rare event.

The results obtained indicate that by means of the system of ionization chambers it is possible not only to measure the size of the showers produced by μ mesons, but also to find the direction of these showers. High-energy μ-meson showers arise primarily from bremsstrahlung photons. In the selection of showers of a given energy the effective energy of the μ meson proved to be very close to the energy of a bremsstrahlung photon whose direction of travel almost coincides with the direction of the μ meson. Therefore, the use of the system of ionization chambers, which allows a good determination of the direction and energy of the μ meson showers, makes it possible to obtain the angular distribution of μ mesons of different energies ($E > 10^{11}$ ev). It is necessary here to place the apparatus under a layer of ground sufficient to absorb the nuclear-active particles (5 – 8 m water equivalent).

The investigation of the angular distribution of showers produced by μ mesons of different energies may give important information as regards the mechanism of the μ-meson generation.[5] The investigation of events involving the simultaneous passage of high-energy μ mesons is also of great interest.

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ON THE EFFECT OF IMPURITIES ON THE ENERGY SPECTRUM OF ELECTRONS IN BISMUTH

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The effect of small additions of Pb on the quantum oscillations of the magnetic susceptibility in Bi was studied at temperatures between 1.7 and 4.2° K and various orientations of the magnetic field relative to the crystallographic axes of the specimens. It was established that the form of the Fermi surface and the electron effective-mass tensor are practically unchanged when the concentration of lead is increased to 0.037 wt%, while the limiting Fermi energy and the concentration of electrons diminish with increasing lead concentration. It was shown that the change in the anisotropy of the magnetic susceptibility of Bi when the content of Pb is increased cannot be explained by a change in the parameters of the electrons and holes; the experimentally observed variations of susceptibility and its anisotropy with the Pb concentration can be fully explained using an idea (due to Adams[3]) that strong diamagnetism exists in the valence band of bismuth.

INTRODUCTION

To explain the marked growth of the magnetic-susceptibility anisotropy $\Delta\chi$ in Bi at helium temperatures, due to lead impurities (see reference 1), Heine[2] used an idea, due to Adams,[3] concerning the possible existence of strong diamagnetism in the valence band of Bi. Heine started from the supposition that on increasing C — the concentration of Pb — the form of the Fermi surface for electrons does not change, and only a decrease in the electron concentration occurs. It will be remembered that the Fermi surface for electrons in bismuth consists of three ellipsoids situated in crystal momentum space in such a way that their smallest axes coincide with the two-fold axes, while the largest axes are inclined 5.5° to the plane perpendicular to the three-fold axis; the ellipsoids go into one another by rotations of 120° around the trigonal axis.[4]

At present, however, there are no experimental data to support Heine's supposition. It is not possible to draw any conclusion on how an increase in the lead concentration deforms the Fermi surface in Bi from the work of Shoenberg and Zakki Uddin,[1] since they only studied the change of frequency E_0/β_i (E_0 is the limiting Fermi energy, β_i is a function of the effective masses and the angle ψ between the direction of the magnetic field **H** and the three-fold axis of the specimen) of the quantum oscillations (and, consequently, also the change of the areas S_m, the external sections of the Fermi surface by planes perpendicular to **H**) for two orientations of **H** in the plane perpendicular to the trigonal axis, which corresponds to the change of the smallest value of S_m and that closest to it.

Since the value of $\Delta\chi$ for pure bismuth is mainly determined by the anisotropies of the three ellipsoids of the electronic part of the Fermi surface,[5] and the contribution to the susceptibility of the hole part of the surface obtained in references 6 and 7 is very small, then it would also be supposed that the increase of $\Delta\chi$ for bismuth is associated with the change of form of the ellipsoids.

We have made a study of the oscillations of the magnetic susceptibility in specimens of Bi with various Pb contents and with various crystallographic orientations of the specimens in the magnetic field, in order to determine the nature of the change in the shape of the Fermi surface and the parameters determining it when the concentration of Pb impurity is changed.

MEASUREMENT METHODS, SPECIMENS

The measurements were made on a torsion balance in a homogeneous magnetic field of strength up to 12 koe at liquid-helium temperatures, with the specimen at a fixed position in the field.[8] The torque N acting on the specimen, was balanced by the torque developed by a compensating device which consisted of a coil mounted on a cylindrical copper damper fixed on the suspension system of the balance. The damper was in the field of a permanent magnet. The coil consisted of 100 turns of PÉ wire of 0.05 mm diameter.

A bronze ribbon suspension with an elastic constant K = 19.4 dyne cm/rad served as one of the current leads to the coil. The fine spiral spring of a type M-25 galvanometer was used for the other lead. The current through the coil was controlled by a KL-48 potentiometer. The position of the suspended system was determined with a mirror system, with accuracy 10—15''.

The alloys of Pb and Bi were made up in Pyrex ampoules, using successive dilution to avoid weighing small quantities of substance. "Hilger" Bi (purity 99.998%) was used, purified by 30-fold recrystallization in vacuum. In the original Bi the electrical resistance fell by a factor of approximately 150 on lowering the temperature from 290 to 4.2° K.

To guarantee the most complete dissolution of the Pb in the Bi, the alloys were held at a temperature of 450° C for 3—4 hours, after which they were rapidly cooled. Seeds from the polycrystalline ingots served (using the method of Kapitza[9]) as primers to grow single-crystal cylinders of length ≈15 mm and diameter about 3.6 mm. In order to decrease the inhomogeneity in impurity distribution, which arises during the growth of a crystal, the specimens were cut off from the primers, attached to them the other way round, and the growing process was repeated.

Specimens about 8 mm long were cut from the central portion of the cylinders and annealed in an atmosphere of helium at 260° C for 5 to 30 days, depending on the Pb concentration. They were then etched in concentrated nitric acid heated to 70—90° C, and their crystallographic orientation was determined on a goniometer with an accuracy of 0.2 to 0.3°.

RESULTS OF THE MEASUREMENTS

Nine specimens of Bi were studied, with Pb contents of 0, 0.012 (two specimens), 0.02 (two specimens), 0.025, 0.03 (two specimens), and 0.037 wt%. The three-fold axis of the specimens was orientated perpendicular to the suspension axis of the balance, whilst the two-fold axis was parallel to it. Measurements were made for various angles ψ between the direction of the trigonal axis and the magnetic field. No measurements were made for the two other main orientations of the crystallographic axes, since they would give no fresh information.

As an example, several curves are given in Fig. 1 showing the variation of the magnetic susceptibility anisotropy $\Delta\chi' = N/H^2 \sin\psi \cos\psi$ on 1/H for the annealed Bi specimens with various Pb contents at a temperature of 1.67° K, and two values of the angle ψ close to $\psi = 0°$ and $\psi = 90°$.

Curves showing the angular variation of the oscillation frequency $E_0/\beta_i \sim S_m$ are given in Fig. 2 for a specimen of the original Bi and specimens of Bi with impurity concentrations of 0.03%. The continuous curves were constructed using Landau's formulae (cited by Shoenberg[4]) and the tensor effective masses proposed for pure Bi by Shoenberg[4] (the dashed curves correspondingly for Bi + 0.03% Pb). As is seen from Fig. 2, the angle of rotation of the Fermi surface ellipsoids about the two-fold axes is practically unchanged on increasing the Pb concentration to 0.03 wt%.

Analysis of the data on the angular variation of E_0/β_i for various Pb concentrations shows that, for all angles, ψ, the impurity causes the frequency of the oscillation, and consequently S_m

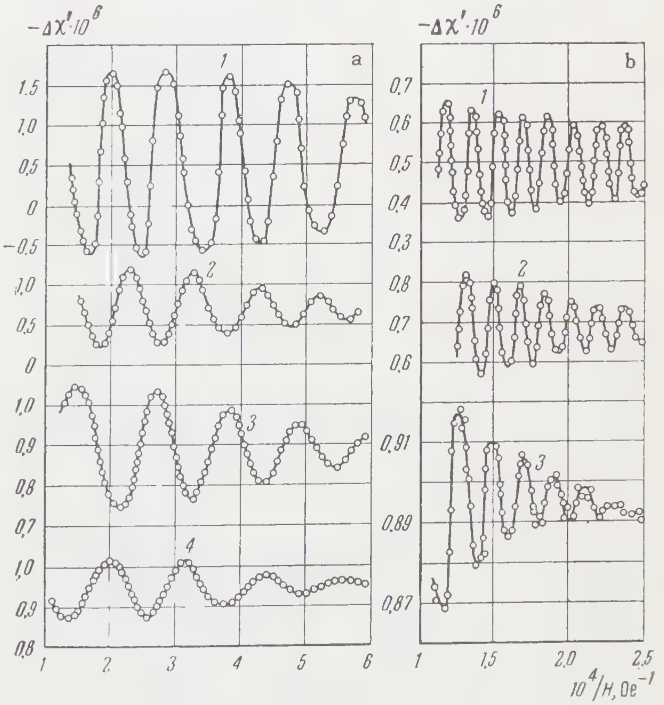


FIG. 1. Variation of magnetic susceptibility anisotropy in Bi-Pb alloys with magnetic field strength for T = 1.65° K: a – for $\psi = +82°$; b – for $\psi = -3°$. Curve 1 – original Bi, Curve 2 – Bi with 0.012% Pb, Curve 3 – Bi with 0.03% Pb, Curve 4 – Bi with 0.037% Pb.

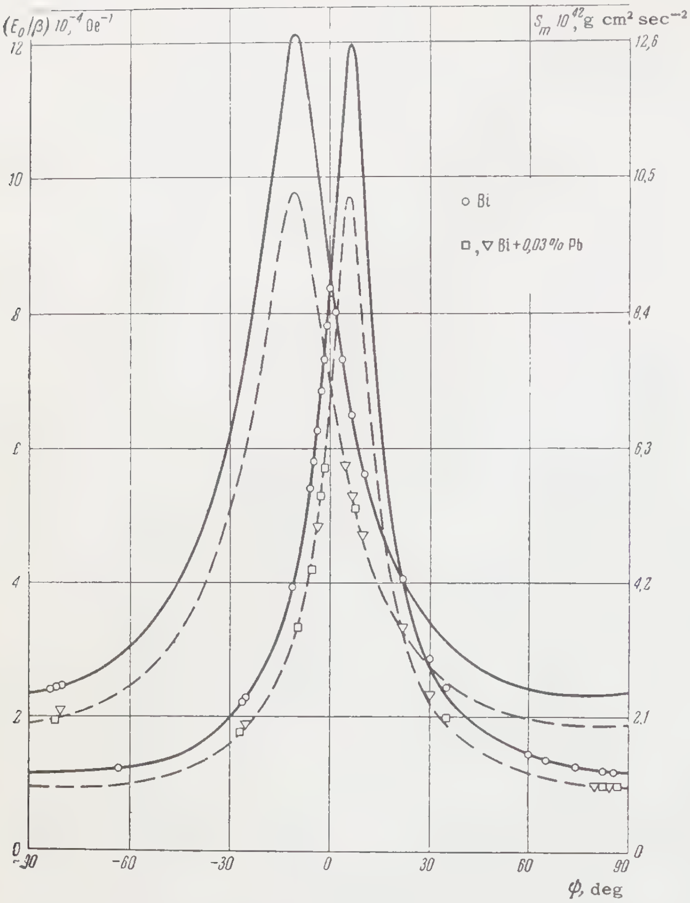


FIG. 2. Variation of oscillation frequency $E_0/\beta_i \approx S_m$ with the angle ψ between the trigonal axis of the specimen and the magnetic field (the two-fold axis was parallel to the suspension axis of the balance).

also, to decrease in the same percentage ratio. The variation of the percentage change of S_m on the concentration of added Pb is given in Fig. 3.

The amplitude of the oscillations decreases strongly with increasing Pb concentration (Fig. 1)—the greatest decrease of amplitude being observed for high-frequency oscillations corresponding to the range of angles close to $\psi = 0°$. This fact greatly hampers the analysis of data obtained for lead concentrations exceeding 0.04 wt%.

The good agreement between the results obtained on two specimens (0.012% Pb), cut from different sides of the original single-crystal cylinder, shows that possible inhomogeneity in the distribution of impurities along the specimens is small, and its effect does not influence the limits of accuracy of the measurements. For two specimens (0.03% Pb) with identical annealing treatments, a larger oscillation amplitude was observed in the specimen for which the original alloy was maintained for three hours at a temerature of 450° C. This result appears to show

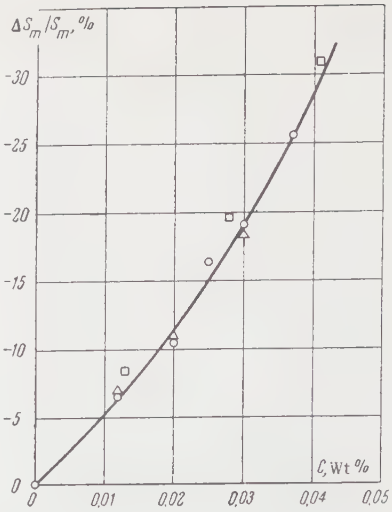


FIG. 3. The change of the extremal sections of the Fermi surface (in %) on increasing the Pb concentration (□ – data of reference 1 for the orientation with **H** perpendicular to the trigonal axis of the specimens).

that an "anneal" of the original alloy diminishes the microinhomogeneity in the impurity distribution. A small discrepancy between the data obtained on specimens with 0.02% Pb concentrations (one specimen was not annealed, the other was annealed for four days) shows that the effect of annealing on the frequency of the oscillations is small, although the oscillation amplitude after annealing increased.

A small increase of the oscillation frequency occurred on lowering the temperature. This effect, absent in pure Bi, increases on increasing the Pb concentration. The maximum observed change of the quantity E_0/β_i on lowering the temperature from 3.2 to 1.6° K in specimens with concentrations 0.012 to 0.037% Pb amounted to 0.7 to 3%.

DISCUSSION OF THE RESULTS

1. Change of the Form of the Fermi Surface for Electrons. The data presented in Fig. 2 show that two of the principal sections (the smallest, S_1, and the intermediate, S_2) of the Fermi surface ellipsoids change in the same percentage ratio. Since the anisotropy of the ellipsoids is principally determined by the anisotropy of these two sections, it can be supposed that the third (the largest) principal section, S_3, changes in the same percentage ratio.

Thus, under the influence of small lead concentrations (up to 0.03%), the Fermi surface for electrons is unchanged in shape. This is, apparently, connected with the fact that Pb atoms, having atomic radius close to Bi atoms, do not essentially change the anisotropy of the Bi lattice. It is interesting to note that the variation of $\Delta S_m/S_m$ (see Fig. 3) with Pb concentration is noticeably nonlinear. Thus, the value of the derivative $\partial(\Delta S_m/S_m)/\partial C$ increases by a factor of approximately 1.5 on increasing C from 0 to 0.035%.

2. Change of the Limiting Fermi Energy for Electrons. To determine E_0^e, the temperature variation of the oscillation amplitude [for angles $\psi = -(15 \text{ to } 27)°$ and $+22°$] was used; knowing this, the value of β_i can be calculated, and then from the known quantity E_0^e/β_i one can also evaluate E_0^e.

From Landau's formulae[4] (on the assumption that the Dingle factor does not depend on temperature), it follows that the ratio of the amplitudes ω_T, at two different temperatures, T_1 and T_2, where $T_2 = 2T_1$ for the same value of the field has the form

$$\omega_{T_1} / \omega_{T_2} = \cosh\,(2\pi^2 kT_1 / \beta_i H) \tag{1}$$

(k is the Boltzmann constant). The variation of E_0^e with Pb concentration calculated in this way is shown in Fig. 4. It is apparent that the variation of E_0^e with C is not linear and that the derivative $\partial E_0^e/\partial C$ increases with increase of C.

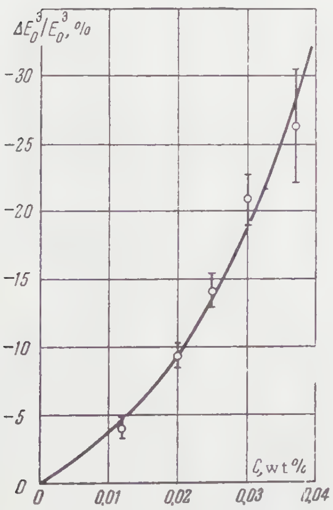


FIG. 4. The change of the limiting Fermi energy for electrons (in %) in Bi as the Pb concentration changes.

It is necessary to bear in mind that the nonlinear variation of E_0^e on C can also be caused by the presence in the original Bi of small concentrations of impurity elements of Group VI (donors), which can compensate part of the added Pb, and by a variation of the Dingle factor[4] x with temperature, which gets greater as C increases. (The Dingle factor x takes into account the change of oscillation amplitude due to scattering of electrons by impurities and irregularities in the lattice.)

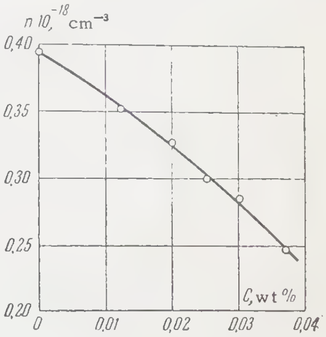
A study of the galvanomagnetic properties of the original Bi at helium temperatures and the treatment of the data obtained by the method given in reference 10 shows, however, that elements of Group IV predominate in the original Bi, and the determination of x from the variation of the oscillation amplitude with magnetic field strength for T = const (which, unfortunately, cannot be done sufficiently accurately) appears to exclude also the second possible cause of nonlinearity in the variation of E_0^e with C.

3. Change of Electron Concentration. The electron concentration can be determined from the formulae

$$n^e = \frac{6V}{(2\pi\hbar)^3}, \qquad V = \frac{4}{3\sqrt{\pi}}(S_1 S_2 S_3)^{1/2}, \tag{2}$$

where V is the volume of one of the Fermi surface ellipsoids. For pure Bi the concentration of electrons n^e, determined from (2), is 0.39×10^{18} cm^{-3}. Using the data of Fig. 2, we obtain the dependence of n^e on C (Fig. 5). The fact that this variation is nearly linear indicates that it is correct to consider Pb impurities at small concentrations as acceptors. However, one must bear in mind here that Pb has an exceptionally low "efficiency" as an acceptor in Bi: to decrease the carrier concentration by one electron requires ~55 atoms of Pb. Therefore, the assumption that one Pb atom changes the electron concentration by unity, which was made by Galt et al,[11] is too crude an approximation.

FIG. 5. Variation of electron concentration n in Bi with the concentration of Pb impurity.



4. Change of Effective Masses. The effective masses m_1^e, m_2^e, and m_3^e, in the system of coordinates related to the principal axes of the Fermi surface ellipsoids, are given by the formulae

$$S_1 = 2\pi E_0 \sqrt{m_1^e m_3^e}, \qquad S_2 = 2\pi E_0 \sqrt{m_1^e m_2^e},$$
$$S_3 = 2\pi E_0 \sqrt{m_2^e m_3^e}. \tag{3}$$

Since,

$$\Delta S_1 / S_1 = \Delta S_2 / S_2 = \Delta S_3 / S_3,$$

it follows from (3) that

$$\Delta m_1^e / m_1^e = \Delta m_2^e / m_2^e = \Delta m_3^e / m_3^e = \Delta S / S - \Delta E_0^e / E_0^e.$$

The first quantities $\Delta S/S$ and $\Delta E_0^e/E_0^e$, as can be seen from Figs. 2 and 3, lie close to one another. Therefore, to a first approximation it can be considered that the effective masses do not change on increasing the Pb concentration from 0 to 0.037%. This result agrees with the data obtained by studying cyclotron resonance in Bi containing impurities.[11]

5. Change of the Constant Part of the Magnetic Susceptibility Anisotropy. The constant part of the magnetic susceptibility anisotropy for electrons has the form

$$\Delta\chi^{e} = \chi^{e}_{\perp} - \chi^{e}_{\parallel} = \frac{e^2 E_0^{1/2}}{4\sqrt{2}\,\pi^2 c^2 \hbar \rho}\;\frac{m_1^{e} + m_2^{e} - 4m_3^{e}}{(m_1^{e} m_2^{e} m_3^{e})^{1/2}}\,. \tag{4}$$

As shown in reference 5, the measured value of the anisotropy* for pure Bi, $\Delta\chi' = \chi'_\perp - \chi'_\parallel = -(1.7 \times 10^{-6} - 1.2 \times 10^{-6}) = -0.5 \times 10^{-6}$ cgs emu at helium temperatures agrees well with the value $\Delta\chi^e$ calculated from formula (4), using the effective mass values determined by Shoenberg:

$$m_1^{e} = 2.4 \cdot 10^{-3} m_0, \qquad m_2^{e} = 2.53 m_0, \qquad m_3^{e} = 0.025 m_0.$$

In fact, calculation gives

$$\Delta\chi_0^{e} = \chi^{e}_{\perp} - \chi^{e}_{\parallel} = -(0.52 \cdot 10^{-6} - 0.02 \cdot 10^{-6}) = -0.5 \cdot 10^{-6}.$$

The anisotropy of the hole susceptibility is negligibly small. Using the hole parameters previously determined:[6,7]

$$m_1^{h} = m_2^{h} = 0.05 m_0, \qquad m_3^{h} = 0.7 m_0,$$
$$E_0^{h} = 2.5 \cdot 10^{-14}\ \text{erg} \qquad (n^{h} = 0.34 \cdot 10^{18}\ \text{cm}^{-3}),$$

we find

$$\chi^{h}_{\perp} = -0.002 \cdot 10^{-6}, \qquad \chi^{h}_{\parallel} = -0.026 \cdot 10^{-6},$$
$$\Delta\chi_0^{h} = -0.024 \cdot 10^{-6}.$$

It is, therefore, of interest to determine, using (4), the change of $\Delta\chi^e$ on increasing the Pb concentration, and to compare the result obtained with the experimentally observed variation of $\Delta\chi'$ with C. The results of this comparison are given in Fig. 6. As is seen from Fig. 6, not only a quantitative, but also a marked qualitative, difference is observed between the curves. Thus, if the value of $\Delta\chi'$ in pure Bi is determined principally by the anisotropy of the electronic part of the Fermi surface, then on increasing the Pb concentration there appears an additional susceptibility anisotropy, which cannot be explained by changes in either the electronic or the hole parts of the Fermi surface.

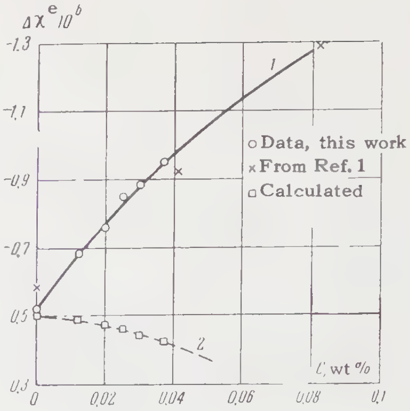


FIG. 6. The variation of the constant part of the magnetic susceptibility anisotropy $\Delta\chi^e$ for Bi with the Pb concentration. Curve 1 – experimental data, Curve 2 – calculated from formula (4).

It is very possible that the results obtained indicate the correctness of a deduction made by Adams[3] about the existence of a large diamagnetism, χ^A, caused by the presence of filled states close to the zone boundaries and unfilled states outside it with small effective masses.

As Adams showed, the value of this additional diamagnetism for a given orientation **H** is given by the formula

$$\chi^{A} \approx (m_0 / \overline{m})\,\chi_0, \tag{5}$$

where $\overline{m}$ is the mean value of the electron effective mass at the boundary of the zone in the plane perpendicular to **H**, and χ_0 is the susceptibility corresponding to the valence band of Bi. The presence of electrons outside the zone and holes inside it causes χ^A to decrease. Since at helium temperatures the experimental values of the diamagnetic susceptibility of Bi in directions perpendicular and parallel to the trigonal axis are $\chi'_\perp = -1.7 \times 10^{-6}$ and $\chi'_\parallel = -1.2 \times 10^{-6}$, but the calculated values are $\chi_\perp = \chi^e_\perp + \chi^h_\perp = -0.522 \times 10^{-6}$ and $\chi_\parallel = \chi^e_\parallel + \chi^h_\parallel = -0.046 \times 10^{-6}$, then the value of χ^A, decreased due to the presence of $0.39 \times 10^{18}\ \text{cm}^{-3}$ electrons and $0.34 \times 10^{18}\ \text{cm}^{-3}$ holes, should be isotropic and equal to $\sim 1.2 \times 10^{-6}$ (the paramagnetism of electrons and holes in Bi can apparently be neglected, owing to their effective mass).

It follows from the experimental data of Shoenberg and Zakki Uddin[1] that on increasing the Pb concentration the value of $\chi'_\perp$ increases to $\sim 2.5 \times 10^{-6}$, goes through a maximu at $C \approx 0.15\%$, and then decreases. The value of $\chi'_\parallel$ decreases monotonically on increasing C; however, this change is a slower one than for $\chi'_\perp$ (see Fig. 7).

Since for $C \approx 0.1\%$ Pb the concentration of electrons should tend to zero (Fig. 5), extrapolation of the falling parts of the curves $\chi'_\perp(C)$ and $\chi'_\parallel(C)$ back to $C = 0$ (the broken curves in Fig. 7) allows the values of $(\chi^A_\perp)'$ and $(\chi^A_\parallel)'$ to be eval-

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*Primes are used, in what follows, to indicate experimental values of the corresponding quantities.

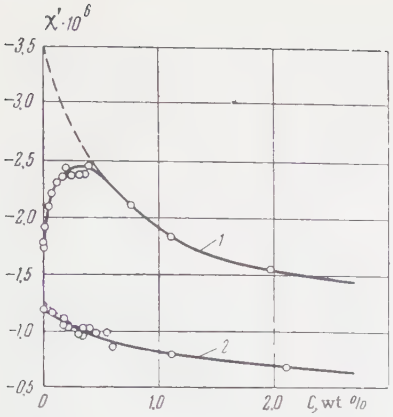

FIG. 7. Variation of $\chi'_\perp$ (curve 1) and $\chi'_{\parallel}$ (curve 2) with Pb concentration according to the data of reference 1.

uated for the valence zone of Bi in the absence of electrons and holes: $(\chi^A_\perp)' \approx 3.5 \times 10^{-6}$, $(\chi^A_{\parallel})' \approx -1.3 \times 10^{-6}$.

It is interesting to note that the values of $(\chi^A_\perp)'$ and $(\chi^A_{\parallel})'$ found in this way agree well with the values of $\chi^A_\perp$ and $\chi^A_{\parallel}$ calculated from formula (5). Writing formula (5) in the form

$$\chi^A_\perp = \chi_0 m_0 / \sqrt{m^e_1 m^e_3} + \chi_0 m_0 / \sqrt{m^h_0 m^h_3},$$

$$\chi^A_{\parallel} = \chi_0 m_0 / \sqrt{m^e_1 m^{e*}_2} + \chi_0 m_0 / m^h_1 \tag{6}$$

and substituting in it the values of the effective masses $m^e_1 = 0.0024 m_0$, $m^{*e}_2 = \frac{1}{2} m^e_2 = 1.26 m_0$, $m^e_3 = 0.025 m_0$, $m^h_1 = m^h_2 = 0.05\ m_0$, $m^h_3 = 0.7 m_0$ and the value $\chi_0 = -0.034 \times 10^{-6}$, calculated* for electrons in the valence band of Bi with effective mass $\sim m_0$ and end-point energy ~7.5 ev, we obtain

$$\chi^A_\perp = -4.3 \cdot 10^{-6} - 0.18 \cdot 10^{-6} = -4.5 \cdot 10^{-6},$$

$$\chi^A_{\parallel} = -0.62 \cdot 10^{-6} - 0.68 \cdot 10^{-6} = -1.3 \cdot 10^{-6}.$$

We remark that Eqs. (6) also describe well the variation of $\chi'_\perp$ and $\chi'_{\parallel}$ on Pb concentration.

The first term of formula (6) provides the main contribution to $\chi^A_\perp$, and therefore the decrease of electron concentration on increasing the Pb concentration causes a pronounced increase of $\chi^A_\perp$, and, consequently also of $\chi'_\perp$. For C ≈ 0.1% the electron zone is completely emptied and the further slower decrease of $\chi'_\perp$ is associated with the increasing concentration of holes in the valence band of Bi.

The electronic contribution to $\chi^A_{\parallel}$ does not have a decisive role, and therefore when the concentration of holes is increased by increasing the Pb concentration the value of $\chi'_{\parallel}$ diminishes.

Thus, if it is assumed that in bismuth, apart from the two groups of electrons and holes mentioned, there are no other carriers possessing large (of the order 1×10^{-6}) diamagnetic susceptibility, the magnetic properties of Bi at helium temperatures and their variation on impurity content can be completely explained on the basis of the work of Heine[2] and Adams.[3]

In conclusion we take the opportunity of thanking A. I. Shal'nikov for his interest in the work, M. I. Kaganov for discussion of the results, and G. A. Kytin for help with the measurements.

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Translated by K. F. Hulme

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*It should be borne in mind that the calculation of χ_0 for the valence band of Bi by Landau's formula, as done in reference 3, is not rigorous, nor is the assumption that χ_0 is isotropic. However, since the values of the effective masses for electrons are not sufficiently accurate, there does not exist at the present time the basis from which to calculate the anisotropy of χ_0 by formula (6).

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THE SPATIAL AND CHARGE PARITIES AND TWO-MESON ANNIHILATION OF THE PROTON-ANTIPROTON SYSTEM

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If the proton and the antiproton are described by the Dirac equation, the system $(p, \bar{p})$ has definite charge and spatial parities. It is proposed that experiments on annihilation, $\bar{p} + p \rightarrow \pi^+ + \pi^-$, $\bar{p} + p \rightarrow \pi^0 + \pi^0$, be used to test this consequence of the Dirac equation.

1. INTRODUCTION

As is well known, if the proton and the antiproton are described by the Dirac equation the proton-antiproton system has quite definite spatial and charge parities.

If the proton has spatial parity ± 1 or ±i, then the parity of the antiproton must be ∓1 or ±i, respectively. Thus the product of the intrinsic parities of p and $\bar{p}$ is -1 (cf. the book by Akhiezer and Berestetskiĭ,[1] Sec. 8, Art. 3, in particular the note on page 55).

Generally speaking, charge conjugation gives the wave function of the charge-conjugate state apart from an arbitrary phase factor:

$$\hat{C}\psi_p = e^{i\eta}\psi_{\bar{p}}, \quad \hat{C}\psi_{\bar{p}} = e^{i\bar{\eta}}\psi_p. \tag{1}$$

In order for the wave function $\psi_{\bar{p}}$ of the antiproton (or the field operator) to obey the Dirac equation, we must have $\bar{\eta} = -\eta$, i.e., $\hat{C}^2$ must be equal to $+1$. Then we get for the complete charge parity of a particle-antiparticle system that is in a state with definite orbital and spin angular momenta l and s the value $(-1)^{l+s}$ (cf. the book by Matthews,[2] Chap. 8, Art. 5, and also the paper by Wolfenstein and Ravenhall[3]).

From the fact that the antiproton exists it does not yet follow that the proton and the antiproton obey the Dirac equation (cf. the review by Segre[4]). We recall that the discovery of positive electrons was immediately followed by a series of experiments for the purpose of establishing their identity with the antiparticles whose existence followed from the correctness of the Dirac equation for the electron (cf. the article by de Benedetti and Corben,[5] page 191). In particular, it has been verified that the polarizations of annihilation γ-ray quanta are perpendicular, which is a simple consequency of the pseudoscalar nature of the s state of the positron and the conservation laws.[5,6]

We are suggesting analogous experiments. Let us suppose that there can be values of the spatial and charge parities of the system $(p, \bar{p})$ different from the values that follow from the Dirac equation.* Then from the law of conservation of angular momentum and from invariance with respect to space reflection and charge conjugation one can get different selection rules for the two-meson annihilation $\bar{p} + p \rightarrow \pi + \pi$, and test them experimentally.

2. THE SELECTION RULES FOR TWO-MESON ANNIHILATION FOR VARIOUS TYPES OF PARITIES OF THE SYSTEM $(p, \bar{p})$

As before, we shall regard $\bar{p}$ as the particle charge-conjugate to the proton. With our purpose of testing the correctness of the Dirac equation we cannot, however, define $\hat{C}$ as the operator that converts the Dirac particle into its antiparticle. The example of K^0 and $\bar{K}^0$ shows that the operator $\hat{C}$ can be given a more general definition, analogous to the definition of the isotopic-spin operator (the operator $\hat{C}$ will convert p into $\bar{p}$ like the operator that converts a proton into a neutron).

We assume that for the charge-conjugate particle the mass (inertial), the momentum (or coordinate), the square of the spin, the spin projection, and, in general, all the mechanical characteristics are just the same as for the original particle (we may regard the spatial parity also as a mechanical characteristic, but this is not obligatory). All the other characteristics, namely the electric charge, the baryon or lepton number, and the strangeness are changed into their opposites by the charge-conjugation operator $\hat{C}$.

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*Suppose, for example, that p and $\bar{p}$ obey two independent two-component equations of the type given by Shirokov,[7] which can describe particles of spin $\frac{1}{2}$ with different (and entirely independent) parities, charges, strangenesses, and so on.

This definition still gives no connection between η and $\bar{\eta}$ in Eq. (1). Both the value of $\hat{C}^2$ when applied to the wave function of p or $\bar{p}$ and the charge parity of the system $(p, \bar{p})$ in a state with definite l and s can be complex numbers of unit absolute value (cf. Appendix A). This is an example of a difference between the operator $\hat{C}^2$ and the square of the reflection operator I. In fact, repeated inversion of the coordinates is an identical transformation, and its representation, i.e., the value of I^2, must be unity for single-valued representations and ±1 for spinor representations. For $\hat{C}$ there is no corresponding fundamental group (such as, for example, a group of coordinate transformations); $\hat{C}$ is defined only through its action on wave functions. Thus after repeated charge conjugation one gets the same wave function only apart from an arbitrary phase factor.

If p and $\bar{p}$ are not described by the Dirac equation, their spatial parities are not related. A particle of half-integral spin can have the spatial parities $+1$, -1, $+i$, $-i$ (reference 2, Chap. 7, Art. 3). These same values of the parity can occur for a system of two particles.

It will be clear from the further argument that from the fact of annihilation of $(p, \bar{p})$ into π mesons it follows that the charge parity of $(p, \bar{p})$ can take only the values $\pm(-1)^{l+s}$, and the intrinsic spatial parity of the π meson only the values ±1. Therefore we shall study four possibilities for the spatial and charge parities of the $(p, \bar{p})$ system: $\{-1, +1\}$ (the ordinary Dirac case), $\{-1, -1\}$, $\{+1, +1\}$, and $\{+1, -1\}$. The first value given is the parity of the π meson, and the second is the coefficient $+1$ or -1 in the expression $\pm 1(-1)^{l+s}$ for the total charge parity of the system $(p, \bar{p})$.

TABLE I

Parity signature	Selection rules for two-meson annihilation	Additional selection rule for $\bar{p}+p \to \pi^0+\pi^0$ (l' is even)
$\{-1, +1\}$	$s=1, l'=l\pm 1$	l — odd
$\{-1, -1\}$	two-meson annihilation is forbidden	
$\{+1, +1\}$	$s=0, \quad l=l'$	l — even
$\{+1, -1\}$	$s=1, \quad l=l'$	l — even

Assuming the usual conservation laws* for the total angular momentum $\mathbf{l} + \mathbf{s} = \mathbf{l}'$ ($\mathbf{l}'$ is the orbital angular momentum of the two mesons), the spatial parity of π: $(-1)^l = (-1)^{l'}$, and the charge parity: $\pm(-1)^{l+s} = (-1)^{l'}$, we get the selection rules shown in Table 1 for two-meson annihilation $\bar{p} + p \to \pi + \pi$ under the various possibilities for the parities.

In principle we can choose between the possibilities by carrying out a phase-shift analysis of the reaction $\bar{p} + p \to \pi + \pi$ (which requires experiments with polarized $\bar{p}$ or p). We shall show that distinctions can be made between the possibilities from the form of the angular distribution and from the energy dependence of the total cross section at small energies of the incident antiprotons.

We emphasize at once that we must exclude from consideration cases with preliminary formation of "protonium" [we use this name for the bound system $(p, \bar{p})$ analogous to positronium]. The angular distribution of the π mesons from the annihilation of "protonium" is always isotropic, and if there indeed is a dependence of the cross section σ on the energy it is determined just by the Coulomb interaction and is the same for all the parity possibilities. In Appendix B it is shown that if annihilation has occurred at a $\bar{p}$ energy larger than 0.25 Mev in the laboratory system the effect of the formation of "protonium" can be neglected.

Furthermore, in many of the proposed experiments annihilation "in flight" must occur in s or p states $(l = 0, 1)$ of the proton-antiproton system. This requires that the quantity kr_0, be small, where $\hbar k$ is the relative momentum of $\bar{p}$ and p in the center-of-mass system and r_0 is the range of the strong (mesonic) interaction of $\bar{p}$ and p [we assume that $r_0 = \hbar/\mu c = 1.4 \times 10^{-13}$ cm, the Compton wavelength of the π meson (see the paper of Rarita and Schwed[8] and the bibliography given there)]. Then in case there is no Coulomb interaction (for example, for the annihilation of an antineutron and a neutron) the matrix element for the transition from the nucleon-antinucleon state with orbital angular momentum l, total spin s, and momentum $\hbar k$ to the $\pi + \pi$ state with orbital angular momentum l' has for $(kr_0)^2 \ll |4l - 2|$ the following basic sort of dependence on k and l (cf. Table 1 in reference 9):

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*The expression $(-1)^{l'}$ for the charge parity of a system of two π mesons can be obtained from the following considerations. Granting that π^0 is charge-even (since it decays into two photons) and assuming that π^0, π^+, and π^- form an isotopic triplet (so that $\pi^{\pm} = \mp 2^{-\frac{1}{2}} D_2 \pi^0 + i 2^{-\frac{1}{2}} D_1 \pi^0$, where D_2 and D_1 are counterclockwise rotations through $\pi/2$ around the 2 and 1 axes in the isotopic space), we find that $\hat{C}\pi_+ = -\pi_-$ and $\hat{C}\pi_- = -\pi_+$, and for the charge parity of the system (π_+, π_-) we get the expression $(-1)^{l'}$ (granting that π mesons obey Bose statistics).

$$R^{l'}_{sl} \sim (kr_0)^{l+1/2}/(l+1)(2l-1)!! \qquad (2)$$

The wave function of the initial state $\overline{p} + p$ (charged particles) differs only slightly from a plane wave if the antiproton energy E is larger than 0.25 Mev.[10] Therefore we may suppose that the dependence (2) holds for $\overline{p} + p \rightarrow \pi + \pi$ over the range from 0.25 Mev to some tens of Mev. More exactly, it can be shown that for the ratios of transition matrix elements with different values of l, in which we shall be interested in what follows, the Coulomb corrections are smaller than 10 percent at E = 0.25 Mev.

From Eq. (2) we get for the energy dependence of the total cross section of the exothermal reaction in the energy range specified the result $\sigma \sim (E^{1/2})^{2l_0-1}$, where l_0 is the lowest orbital angular momentum for the given hypothesis about the parities, which predominates over all larger values in virtue of Eq. (2) if $(kr_0)^2 \ll |4l - 2|$ (cf. the previously mentioned paper by Wigner,[9] Sec. IIIA, "neutral case." We note that for $E < 0.025$ Mev we have the "charged case," and then $\sigma \sim 1/E$ independent of the parities).

In estimating the contribution of higher angular momenta one must also take into account the coefficients with which the quantities $R^{l'}_{sl}$ occur in the expressions for the differential and total cross sections (cf. Appendix B).

3. PROPOSED EXPERIMENTS

The first problem is to establish the existence or nonexistence of the two-meson annihilation $\overline{p} + p \rightarrow \pi + \pi$ at any energy of the $\overline{p}$ (including the case with preliminary formation of "protonium"). If it does not exist, the parities are those for $\{-1, -1\}$. On the other hand if we suppose that the existence of two-meson annihilation* of proton and antiproton gets established experimentally, then three of our possibilities for the parities remain. Under suitable limitations on the energy of $\overline{p}$ the annihilation cross section will be mainly determined by the elements $R^{l'}_{sl}$ of the transition matrix with the smallest values of l. Assuming further that the matrix elements $R^{l'}_{sl}$ behave essentially as shown in Eq. (2), we get the forms shown in Table II for the energy dependence of the total cross section for two-meson annihilation with the various parity possibilities, correct to deviations of not more than 10 percent within the energy ranges indicated in the table (meaning in all cases the kinetic energy of the antiproton in the laboratory system).

TABLE II

Parity signature	$\overline{p}+p \rightarrow \pi^+ + \pi^-$		$\overline{p}+p \rightarrow \pi^0 + \pi^0$	
	Total cross section $\sigma_+(E)$	Energy, Mev	Total cross section $\sigma_0(E)$	Energy, Mev
$\{-1, +1\}$	$\sigma \infty E^{-1/2}$	$0.5 \div 10$	$\sigma \infty E^{1/2}$	0.5 -- 50
$\{+1, +1\}$	$\sigma \infty E^{-1/2}$	$0.5 \div 10$	$\sigma \infty E^{-1/2}$	0.5 — 50
$\{+1, -1\}$	$\sigma \infty E^{1/2}$	$0.5 \div 40$	$\sigma \infty E^{3/2}$	0.5 — 100

Other possibilities for distinguishing between the parity possibilities are provided by the angular distribution of two-meson annihilations "in flight." It is shown in Appendix C that for $\{+1, -1\}$ the angular distribution $\sigma_+(\vartheta)$ of the reaction $\overline{p} + p \rightarrow \pi^+ + \pi^-$ at any energy must be of the form $\sin^2\vartheta$ in certain ranges of angles near 0° and 180°. The larger the energy of the antiproton, the smaller the range of angles. In particular, for $0.25 < E \leq 10$ Mev the "combined" angular distribution $\sigma_+(\vartheta) + \sigma_+(\pi - \vartheta)$ is of the form $\sin^2\vartheta$ at all angles to within maximum deviations of the order of 10 percent (at angles ϑ close to 90°). For $0.25 < E \leq 40$ Mev it is of the form $\sin^2\vartheta$ in the range 0° — 30° (with the same deviation at angles close to 30°). The lower limit of the energy can of course also be larger than 0.25 Mev.

For this same parity type the angular distribution $\sigma_0(\vartheta)$ of the reaction $\overline{p} + p \rightarrow \pi^0 + \pi^0$ must be of the form $\sin^2\vartheta \cos^2\vartheta$ for angles close to 0° and 90°. In particular, $\sigma_0(\vartheta) \sim \sin^2\vartheta \cos^2\vartheta$ (to accuracy 10 percent) at all angles for $0.25 < E \leq 20$ Mev, and in the ranges 0° — 30°, 60° — 90° for $0.25 < E \leq 50$ Mev (to the same accuracy).

These features of the energy dependence and angular dependence of the cross section enable us to indicate some possible series of experiments for distinguishing between the three parity possibilities.

1. One measures $\sigma_+(\vartheta)$ for $E < 40$ Mev. If it does not go to zero at $\vartheta = 0°$, the possibility $\{+1, -1\}$ can be rejected. To distinguish between $\{-1, +1\}$ and $\{+1, +1\}$ one has only to determine whether $\sigma_0(E)$ decreases or increases as the energy is decreased.* For example, one can measure and compare the mean cross sections for the cases with antiproton energies $10 \leq E < 30$ Mev and those with $30 < E < 50$ Mev.

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*More than 3000 $\overline{p}p$ annihilations are now known; among them there has not been registered a single case of two-meson annihilation.[11] Further investigations are necessary to establish whether this is due to the Dirac parity type $\{-1, -1\}$ or to other causes.

*In the case $\{+1, +1\}$ the cross section $\sigma_0(\vartheta)$ must be isotropic as long as we can neglect R^0_{02}. But this fact cannot be used to distinguish between the two cases, since even for $E \leq 5$ Mev the deviation from isotropy can reach 15 percent; furthermore in the Dirac case $\sigma_0(\vartheta)$ can be accidentally isotropic (cf. Appendix C).

The case in which $\sigma_+(\vartheta)$ is small near 0° requires special examination. This is also possible for the parity types $\{-1, +1\}$ and $\{+1, +1\}$, but only for a certain quite definite relation between the elements $R^1_{0,1}$, $R^0_{1.1}$, $R^2_{1,1}$, . . . and $R^0_{0,0}$, $R^0_{1,0}$, . . . , which can occur accidentally. If, however, $E < 10$ Mev, this relation is in contradiction with the relations (2). Thus to be sure about the legitimacy of the type $\{+1, -1\}$ one has, strictly speaking, still to see whether $\sigma_+(0°)$ remains small for $0.25 < E < 10$ Mev. As can be seen from Table II, this experiment can be replaced by a measurement of $\sigma_+(E)$ in the range 0.5 — 10 Mev. If it decreases with decreasing E, then we have the case $\{+1, -1\}$; if it increases, there remain the other two cases [which can be distinguished by means of the behavior of $\sigma_0(E)$, see above]. But this experiment is evidently more difficult than the measurement of $\sigma_+(\vartheta)$ for $0.25 < E \leq 10$ Mev.

2. One measures $\sigma_0(\vartheta)$ for $E \leq 50$ Mev. If it does not go to zero at 0° and 90°, the possibility $\{+1, -1\}$ can be rejected, and the other two possibilities are distinguished by means of $\sigma_0(E)$, as indicated above.

If $\sigma_0(0°)$ and $\sigma_0(90°)$ are small and this situation holds also for $E < 30$ Mev, the possibility $\{+1, -1\}$ is the correct one. For the other two possibilities the vanishing of $\sigma_0(\vartheta)$ at both 0° and 90° would require the existence of an even more special relation between the $R^{l'}_{sl}$, which is contradicted by Eq. (2) for $E < 30$ Mev (elements would have to be equal that differ by an order of magnitude according to Eq. (2).

3. One measures $\sigma_0(E)$ with enough accuracy to distinguish between the $E^{1/2}$ and $E^{3/2}$ laws over the range 0.5 — 50 Mev in the case of decrease of $\sigma_0(E)$ with decrease of the energy. As can be seen from Table II, this suffices to distinguish between all three possibilities.

We repeat that all of these arguments are based on the assumption that a number of conservation laws hold (cf. Sec. 2) and that the relations (2) are valid. If the cross section does not behave as required by the Dirac case, then one could assume, for example, that parity is not conserved in the annihilation. This, however, would be a deduction of no less fundamental importance than the inapplicability of the Dirac equation to p and $\overline{\text{p}}$.

APPENDIX A

The Charge Parity of $(\text{p}, \overline{\text{p}})$. In the representation of the spin components m_1, m_2, the signs of charge ϵ_1 and ϵ_2, and the angles of the relative momentum the correct antisymmetric wave function of the system $(\overline{\text{p}}, \text{p})$ must be of the form

$$\psi = 2^{-1/2}\{\langle \mathbf{n} m_1 m_2 \varepsilon_1 \varepsilon_2 \;\; l\mu s m\rangle - \langle -\mathbf{n} m_2 m_1 \varepsilon_2 \varepsilon_1 \,|\, l\mu s m\rangle\},$$

where μ and m are the projections of the orbital angular momentum and the total spin. The interchange of the indices on ϵ that characterize state and antistate, which is required for the antisymmetrization, can be accomplished by means of the operator $\hat{C}$:

$$\hat{C}\,\langle \varepsilon_1 \varepsilon_2 \,|\,\rangle = \langle \varepsilon_2 \varepsilon_1 \,|\,\rangle\, e^{i\alpha}.$$

Here $\alpha = \eta + \overline{\eta}$ [cf. Eq. (1)]; $\alpha = 0$ if p and $\overline{\text{p}}$ are described by the Dirac equation. Taking this fact into account we have

$$\langle -\mathbf{n} m_2 m_1 \varepsilon_2 \varepsilon_1 \,|\, l\mu s m\rangle \equiv N Y_{l\mu}(-\mathbf{n})\, C^{sm}_{1/2 m_2 1/2 m_1}\, \langle \varepsilon_2 \varepsilon_1 \,|\,\rangle$$

$$= N(-1)^{l+s-1}\, Y_{l\mu}(\mathbf{n})\, C^{sm}_{1/2 m_1 1/2 m_2}\, \hat{C}\,\langle \varepsilon_1 \varepsilon_2 \,|\,\rangle\, e^{-i\alpha}$$

$$= (-1)^{l+s-1}\,\hat{C}\,\langle \mathbf{n} m_1 m_2 \varepsilon_1 \varepsilon_2 \,|\, l\mu s m\rangle\, e^{-i\alpha},$$

$$\psi = 2^{-1/2}[1 + (-1)^{l+s}\hat{C}e^{-i\alpha}]\,\langle \mathbf{n} m_1 m_2 \varepsilon_1 \varepsilon_2 \,|\, l\mu s m\rangle,$$

from which it follows that the charge parity of the system $(\text{p}, \overline{\text{p}})$ with definite l and s is $(-1)^{l+s}e^{i\alpha}$:

$$C\psi = 2^{-1/2}[\hat{C} + (-1)^{l+s}e^{2i\alpha}e^{-i\alpha}]\,|\, l\mu s m\rangle = (-1)^{l+s}e^{i\alpha}\psi.$$

APPENDIX B

The process of the formation of "protonium" in the collision of a fast antiproton with a proton is the inverse of the photoelectric effect $\gamma + (\text{p}, \overline{\text{p}}) \rightarrow \text{p} + \overline{\text{p}}$. Using the well known expression for the cross section for the photoelectric effect[12] and the principle of detailed balancing, we find that for energies of $\overline{\text{p}}$ much larger than 25 kev (in the l. s.) electromagnetic capture leads to the formation of "protonium" mainly in the 1s state with a total cross section that falls off rapidly with increase of the energy of the $\overline{\text{p}}$:

$$\sigma = \frac{2^7\pi}{3\cdot 137^3}\,\frac{a_0^2}{(ka_0)^5}\,\frac{\xi}{2},$$

where $a_0 = \hbar^2/Me^2 = 0.58 \times 10^{-11}$ cm; $k = \hbar^{-1} \times (ME/2)^{1/2}$, where E is the kinetic energy in the l.s.; $\xi = 1/2$ if the "protonium" is formed in the singlet state, and $\xi = 3/2$ if it is in the triplet state.

For $E = 0.25$ Mev we get for σ a value of the order of 0.01 mb. On the other hand it can be expected that the cross section for two-meson annihilation owing to the strong (mesonic) interaction is of the order of several millibarns at low energies (if it is not forbidden).[13]

APPENDIX C

1. The general formula for the cross section for two-meson annihilation is (cf. e. g., the review article by Blatt and Biedenharn[14]):

$$\sigma(\vartheta)\sim\sum_{m_a,m_b}\Big|\sum_{l'\mu' lsm}Y_{l'\mu'}(\vartheta,\varphi)R^{l'}_{sl}C^{l'\mu'}_{sml0}C^{sm}_{{}^1\!/_2m_a{}^1\!/_2m_b}\sqrt{2l+1}\Big|^2$$

$$=\sum_{s,m}\Big[\sum_{l_1',l_1}Y_{l_1'm}(\vartheta,0)C^{l_1'm}_{sml_10}\sqrt{2l_1+1}R^{l_1'}_{sl_1}\Big]$$

$$\times\Big[\sum_{l_2'l_2}Y_{l_2'm}(\vartheta,0)C^{l_2'm}_{sml_20}\sqrt{2l_2+1}R^{l_2'}_{sl_2}\Big]^*. \qquad \text{(C.1)}$$

The z axis is chosen parallel to the momentum $\mathbf{p}_0$ of the antiproton; m_a and m_b are the spin projections of $\overline{p}$ and p; and $C^{\cdot\cdot}_{\cdot\cdot}$ are Clebsch-Gordan coefficients.

2. If we confine ourselves only to the matrix elements R^0_{11} and R^2_{11} we get from Eq. (C.1) for the reaction $\overline{p}+p\rightarrow\pi^0+\pi^0$ in the "usual" parity possibility ($s=1$, l odd) the result

$$\sigma_0(\vartheta)\sim|R^0_{11}|^2+\frac{5}{2}|R^2_{11}|^2+\sqrt{10}\,\mathrm{Re}\,(R^0_{11}R^{2*}_{11})$$

$$+\cos^2\vartheta\Big[\frac{15}{2}|R^2_{11}|^2-3\sqrt{10}\,\mathrm{Re}\,(R^0_{11}R^{2*}_{11})\Big]\equiv A+B\cos^2\vartheta.$$

Regarding R^0_{11} and R^2_{11} we know only that $|R^0_{11}|^2+|R^2_{11}|^2\leq 1$ (the unitarity relation). For the quantity A we have $A=|R^0_{11}+(5/2)^{1/2}R^2_{11}|^2\geq 0$; B can also be negative (but $A+B\geq 0$), and in particular can be equal to zero (isotropy).

With inclusion of $l=1, 3$ the total cross section is given by

$$\sigma_0=\frac{1}{16}k^{-2}\{|R^0_{11}|^2+5|R^2_{11}|^2+5|R^2_{13}|^2+9|R^4_{13}|^2\}.$$

At E = 50 Mev we have $(kr_0)^2=0.9$. Then the condition of applicability of Eq. (2), $(kr_0)^2\ll|4l-2|$, is well satisfied for R^2_{13}, R^4_{13}, and poorly for R^0_{11}, R^2_{11}. Using nevertheless the estimate (2), we find

$$(5+9)|R_{13}|^2/(1+5)|R_{11}|^2\approx 0.2\%,$$

where the error of this estimate itself is probably 100 percent. Thus we can suppose that $\sigma\sim k^{-2}\times|R_{11}|^2\sim E^{1/2}$ in the range 0.5 – 50 Mev, with accuracy not worse than 0.5 percent.

3. For the case $\{+1,-1\}$ we have $s=1$ and $l=l'$. In view of the fact that $C^{l0}_{10l0}=0$, the sum over m reduces to the sum of equal terms with $m=\pm 1$:

$$\sigma(\vartheta)\sim\Big|\sum_l Y_{l,1}(\vartheta,0)C^{l1}_{11l0}\sqrt{2l+1}R^l_{1l}\Big|^2. \qquad \text{(C.2)}$$

The spherical function $Y_{l,1}(\vartheta,0)$ is proportional to the associated Legendre polynomial of the first kind. Therefore

$$Y_{l,1}(\vartheta,0)\sim(1-\cos^2\vartheta)^{1/2}dP_l(\cos\vartheta)/d\cos\vartheta,$$

$$\sigma_+(\vartheta)\sim\sin^2\vartheta\{|R^1_{11}|^2+\ldots\} \qquad \text{(C.3)}$$

In particular, if we can neglect all the elements R^l_{1l} with $l>2$, then

$$\sigma_+(\vartheta)\sim\sin^2\vartheta\Big\{|R^1_{11}|^2+(10/\sqrt{3})\,\mathrm{Re}\,(R^1_{11}R^{2*}_{12})\cos\vartheta$$

$$+\frac{25}{3}|R^2_{12}|^2\cos^2\vartheta\Big\},$$

$$\sigma_+(\vartheta)+\sigma_+(\pi-\vartheta)\sim\sin^2\vartheta\Big\{|R^1_{11}|^2+\frac{25}{3}|R^2_{12}|^2\cos^2\vartheta\Big\}.$$

For E = 10 Mev we get $kr_0=0.245$ and $(25/3)|\times|R^2_{12}|^2/|R^1_{11}|^2\approx 7$ percent.

For the reaction $\overline{p}+p\rightarrow\pi^0+\pi^0$ in this parity signature l must be even. At any energy the cross section is a linear combination of products of pairs of functions $Y_{21}(\vartheta,0)$, $Y_{41}(\vartheta,0)$, $Y_{61}(\vartheta,0)$, etc. They are all proportional to $\sin\vartheta\,dP_l(\cos\vartheta)/d\cos\vartheta$, and since for even l the Legendre polynomial $P_l(\cos\vartheta)$ depends only on even powers of $\cos\vartheta$, for practical purposes $Y_{l,1}(\vartheta,0)\sim\sin\vartheta\cos\vartheta$, so that

$$\sigma_0(\vartheta)\sim\sin^2\vartheta\cos^2\vartheta\{|R^2_{12}|^2+\ldots\}.$$

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THE SCATTERING MATRIX WITH THE PAULI INTERACTION TAKEN INTO ACCOUNT

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A ten-dimensional formulation of electromagnetic field theory, making use of first-order equations, is employed. Within the framework of this formalism, a general expression is derived for the quantum-electrodynamic scattering matrix taking into account the so-called Pauli interaction without introducing scalar photons. The free field longitudinal photon states are used in order to simplify the calculations of the transition probabilities for unpolarized light.

In an earlier paper by the present authors[1] a ten-dimensional form of the quantum theory of the electromagnetic field in vacuum was developed on the basis of the "generalized" first-order wave equation[2]

$$(\gamma_\rho \nabla_\rho + \gamma_0)\,\psi(x) = 0 \quad .$$

The wave function ψ consists of the components of the potential ψ_μ and of the field $\psi_{\mu\nu} = -\psi_{\nu\mu}$; γ_ρ are the ten-dimensional matrices of the vector meson theory satisfying the Duffin-Kemmer algebra, while γ_0 is the projection matrix:

$$\gamma_0 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{pmatrix}. \tag{1}$$

The advantage of this approach is that scalar photons are automatically eliminated from the theory, and one does not have to impose on the wave function any restrictions associated with the Lorentz condition. States of right- and left-handed circular polarization appear naturally in the theory as the fundamental states of the photon. The second-quantized wave function of the electromagnetic field ψ and the conjugate function $\bar{\psi}$ have the form

$$\psi(x) = \frac{\sqrt{\hbar c}}{2\pi}\int \frac{d^3k}{\sqrt{k_0}} \sum_{q=0,\pm 1} [c_q(\mathbf{k})\,\psi^{(q)}(k)\,e^{ikx} + c_q^{+}(\mathbf{k})\,\psi^{(q)}(-k)\,e^{-ikx}],$$

$$\bar{\psi}(x) = \frac{\sqrt{\hbar c}}{2\pi}\int \frac{d^3k}{\sqrt{k_0}} \sum_{q=0,\pm 1} [c_q^{+}(\mathbf{k})\,\bar{\psi}^{(q)}(k)\,e^{-ikx} + c_q(\mathbf{k})\,\bar{\psi}^{(q)}(-k)\,e^{ikx}]. \tag{2}$$

Here the operators $c_q(\mathbf{k})$ and $\overset{+}{c}_q(\mathbf{k})$ satisfy the commutation relations

$$[c_q(\mathbf{k}), c_{q'}^{+}(\mathbf{k}')]_- = \delta_{qq'}\,\delta(\mathbf{k}-\mathbf{k}'),$$

$$[c_q(\mathbf{k}), c_{q'}(\mathbf{k}')]_- = [c_q^{+}(\mathbf{k}), c_{q'}^{+}(\mathbf{k}')]_- = 0, \qquad q, q' = 0, \pm 1. \tag{3}$$

The quantities $\psi^{(q)}(\pm k)$ are defined by the following formulas (cf. reference 1)

$$\psi^{(\pm 1)}(\pm k) = \begin{pmatrix} 0 \\ \mathbf{e} \\ \pm i\,[\mathbf{k}\times\mathbf{e}] \\ \mp i k_4 \mathbf{e} \end{pmatrix}, \quad \psi^{(\mp 1)}(\pm k) = \begin{pmatrix} 0 \\ \mathbf{e}^* \\ \pm i\,[\mathbf{k}\times\mathbf{e}^*] \\ \mp i k_4 \mathbf{e}^* \end{pmatrix},$$

$$\psi^{(0)}(\pm k) = \begin{pmatrix} k_4 \\ \mathbf{k} \\ 0 \\ 0 \end{pmatrix}, \tag{4}$$

$$\bar{\psi} = \psi^* \eta,$$

where η is the matrix of the bilinear invariant form[1]:

$$\eta = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & -I & 0 \\ 0 & 0 & 0 & I \end{pmatrix}. \tag{6}$$

All the components of the vectors $\psi(k)$, with the exception of the first, are three-dimensional vectors in the notation of (4); correspondingly in the matrices γ_0 and η all the elements with the exception of the first row and the first column are three-dimensional square matrices; $k = (\mathbf{k}, k_4)$ is the four-momentum of the photon, $\mathbf{e}$ and $\mathbf{e}^*$ are three-dimensional complex "circular" vectors having the following properties

$$[\mathbf{k}\times\mathbf{e}] = i\,|\mathbf{k}|\,\mathbf{e}, \quad [\mathbf{k}\times\mathbf{e}^*] = -i\,|\mathbf{k}|\,\mathbf{e}^*, \qquad |\mathbf{e}| = 1.$$

We shall need the commutation relations for the transverse part of $\psi(x)$. By utilizing the methods employed previously[3] and by taking into account the fact that the truncated minimal polynomial of the operator $B(\mathbf{k}) = ik_\rho\gamma_\rho + \gamma_0$ for the transverse part of the electromagnetic field has the form $S^2(\mathbf{k})\,[B(\mathbf{k}) - 1]^2$, we obtain

$$[\psi'(x')\cdot\overline{\psi}'(x'')]_- = \frac{2\hbar c}{(2\pi)^2 i}\int S^2(\mathbf{k})\,(B(k)-1)^2\,\frac{\sin k_0x_0}{k_0}\,e^{i\mathbf{kx}}d^3k,$$

$$x = x' - x''. \qquad (7)$$

Here $S(\mathbf{k}) = ik_a\,|\,\mathbf{k}\,|^{-1}\delta_{abc}\gamma_b\gamma_c$ is the projection operator giving the component of the spin along the direction of motion;[1] δ_{abc} is the Levi-Civita symbol; a, b, c = 1, 2, 3; the prime indicates that only the transverse part of the function $\psi(x)$ is taken.

In order to obtain functions that are solutions of the Dirac equation, we shall use the general method described in reference 3. The Dirac equation for the plane wave $\varphi(x) = \varphi(p)\exp(ipx)$ takes the form

$$(i\hat{p} + \varkappa)\,\varphi(x) = 0, \qquad (8)$$

where $\hat{p} = p_\mu\beta_\mu$; β_μ are the Dirac matrices. The projection operator giving the component of spin along the direction of motion of the particle which commutes with $\hat{p}$ has the form

$$\sigma = i\delta_{abc}p_a\beta_b\beta_c/4\,|\,\mathbf{p}\,|. \qquad (9)$$

The minimal equations for the matrices $i\hat{p}$ and σ can be written in the following form:

$$(i\hat{p}+\varkappa)(i\hat{p}-\varkappa) = 0, \quad (\sigma + {}^1\!/_2)(\sigma - {}^1\!/_2) = 0. \qquad (10)$$

Therefore, in accordance with reference 3, the projection operators corresponding to the four linearly independent solutions of equation (8) have the following form

$$\tau^{(1)}_{\pm}(p) = ({}^1\!/_2 + \sigma)(\varkappa \mp i\hat{p})\,/\,2\varkappa = \pm\,\varphi^{(1)}(\pm p)\cdot\overline{\varphi}^{(1)}(\pm p), \qquad (11)$$

$$\tau^{(2)}_{\pm}(p) = ({}^1\!/_2 - \sigma)(\varkappa \mp i\hat{p})\,/\,2\varkappa = \pm\,\varphi^{(2)}(\pm p)\cdot\overline{\varphi}^{(2)}(\pm p), \qquad (12)$$

where $\overline{\varphi} = \varphi^*\beta_4$. Expressions of the form $\varphi(p)\cdot\overline{\varphi}(p)$ denote the dyads $(\varphi\cdot\overline{\varphi})_{ik} = \varphi_i\overline{\varphi}_k$. The vectors $\varphi^{(1)}(p)$, $\varphi^{(3)}(p)$ are amplitudes of the plane wave $\varphi(x) = \varphi(p)\exp(ipx)$ which describes a free electron of four-momentum p, and correspond to the two possible components of spin along the direction of motion. Similarly, the vectors $\varphi^{(1)}(-p)$, $\varphi^{(2)}(-p)$ are the amplitudes of the plane wave $\varphi(x) = \varphi(-p)\exp(-ipx)$ which according to Feynman[4] can be used to describe a free positron of the same four-momentum and polarization. The following normalization holds[3] for the vectors $\varphi^{(i)}(\pm p)$

$$\varphi^{(i)}(\pm p)\,\overline{\varphi}^{(k)}(\pm p) = \pm\,\delta_{ik}, \qquad (13)$$

as can be easily verified directly by multiplying the operators $\tau(p)$ when i ≠ k, and by evaluating their traces when i = k.

The general solution of the Dirac equation can be represented as usual in the form of the three-dimensional Fourier integral (cf., for example, reference 5)

$$\varphi(x) = (2\pi)^{-3/2}\int\sqrt{\frac{\varkappa}{p_0}}\sum_{r=1,2}[a_r(\mathbf{p})\,\varphi^{(r)}(p)\,e^{ipx}$$

$$+\,b_r^+(\mathbf{p})\,\varphi^{(r)}(-p)\,e^{-ipx}]d^3p,$$

$$\overline{\varphi}(x) = (2\pi)^{-3/2}\int\sqrt{\frac{\varkappa}{p_0}}\sum_{r=1,2}[a_r^+(\mathbf{p})\overline{\varphi}^{(r)}(p)\,e^{-ipx}$$

$$+\,b_r(\mathbf{p})\,\overline{\varphi}^{(r)}(-p)\,e^{ipx}]\,d^3p, \qquad (14)$$

where

$$[a_r(\mathbf{p}), a_{r'}^+(\mathbf{p}')]_+ = \delta_{rr'}\delta(\mathbf{p}-\mathbf{p}'),$$

$$[b_r(\mathbf{p}), b_{r'}^+(\mathbf{p}')]_+ = \delta_{rr'}\delta(\mathbf{p}-\mathbf{p}'), \qquad (15)$$

and all the other anticommutators vanish.

The general relation obtained in reference 3 immediately yields, with the aid of (11) and (12), the well-known commutation relations for $\varphi(x)$ and $\overline{\varphi}(x)$:

$$[\varphi(x')\cdot\overline{\varphi}(x'')]_+ = \frac{i}{(2\pi)^3}(\beta_\mu\nabla_\mu - \varkappa)\int\frac{d^3p}{p_0}\,e^{i\mathbf{px}}\,\sin p_0x_0,$$

$$x' - x'' = x. \qquad (16)$$

With the aid of the same formulas (11) and (12) we can very simply obtain the pairing of the operators $\varphi(x)$ and $\overline{\varphi}(x)$. By utilizing the dyad notation we can write this relation in the form

$$\underline{\varphi(x')\cdot\overline{\varphi}(x'')} = S^F(x'-x'') = T\,[\varphi(x')\cdot\overline{\varphi}(x'')]$$

$$-\,N\,[\varphi(x')\cdot\overline{\varphi}(x'')], \qquad (17)$$

where T and N are the symbols denoting the mixed and the normal products respectively, S^F is the matrix operator of the pairing. On substituting (14) into (17) and taking (11) and (12) into account we easily obtain the well-known expression

$$S^F(x) = \frac{1}{2\,(2\pi)^3}(\varkappa - \beta_\mu\nabla_\mu)\int e^{i\mathbf{px}}\,e^{-ip_0|x_0|}\,\frac{1}{p_0}\,d^3p. \qquad (18)$$

The ten-dimensional representation for the electromagnetic field functions enables us to generalize the Lagrangian in a natural manner to the case of the interaction of the Pauli type. As is well known, the usual Lagrangian for the interaction between the electromagnetic and the Dirac fields has the form

$$L^{(1)}_{\text{int}} = -\,ie\overline{\varphi}(x)\,\psi_\mu(x)\,\beta_\mu\varphi(x), \qquad (19)$$

while the interaction Lagrangian of the Pauli type has the form

$$L^{(2)}_{\text{int}} = \tfrac{1}{2}\,iel\overline{\varphi}(x)\,\psi_{\mu\nu}(x)\,\beta_\mu\beta_\nu\varphi(x), \qquad (20)$$

where l is a real constant having the dimension of length.[6]

We introduce the ten-dimensional matrix vector

$$\Gamma = (\beta_4, \beta_1, \beta_2, \beta_3, \beta_2\beta_3, \beta_3\beta_1, \beta_1\beta_2, \beta_1\beta_4, \beta_2\beta_4, \beta_3\beta_4). \quad (21)$$

Then, by combining (19) and (20) we can write*

$$L_{\text{int}} = L_{\text{int}}^{(1)} + L_{\text{int}}^{(2)} = -ie\bar{\varphi}(x)\left[\Gamma(1-\gamma_0-l\gamma_0)\psi(x)\right]\varphi(x). \quad (22)$$

By starting with the fact that the Lagrangian of the free electromagnetic field has in the ten-dimensional formulation the form[1,2] $L_{ph} = \psi(x)(\gamma_\rho \nabla_\rho + \gamma_0)\psi(x)/8\pi$ we can construct the Lagrangian for the interacting electromagnetic and electron-positron fields:†

$$\begin{aligned} L(x) = {} & (8\pi)^{-1}\bar{\psi}(x)(\gamma_\rho\nabla_\rho + \gamma_0)\psi(x) + \hbar c\bar{\varphi}(x)(\beta_\mu\nabla_\mu + \varkappa)\varphi(x) \\ & -\tfrac{1}{2}ie\bar{\varphi}(x)\Gamma(1-\gamma_0-l\gamma_0)\psi(x)\varphi(x) \\ & -\tfrac{1}{2}ie\bar{\varphi}(x)\bar{\psi}(x)(1-\gamma_0+l\gamma_0)\Gamma\varphi(x). \end{aligned} \quad (23)$$

In the classical case, we obtain from (23) by variation the following system of equations for the coupled fields:

$$(\gamma_\mu\nabla_\mu + \gamma_0)\psi(x) = 4\pi ie\bar{\varphi}(x)(1-\gamma_0+l\gamma_0)\Gamma\varphi(x), \quad (24)$$

$$(-\tilde{\gamma}_\mu\nabla_\mu + \gamma_0)\bar{\psi}(x) = 4\pi ie\bar{\varphi}(x)\Gamma(1-\gamma_0-l\gamma_0)\varphi(x), \quad (25)$$

$$(\beta_\mu\nabla_\mu + \varkappa)\varphi(x) = (ie/\hbar c)\Gamma(1-\gamma_0-l\gamma_0)\psi(x)\varphi(x), \quad (26)$$

$$(\tilde{\beta}_\mu\nabla_\mu - \varkappa)\bar{\varphi}(x) = -(ie/\hbar c)\Gamma(1-\gamma_0-l\gamma_0)\psi(x)\bar{\varphi}(x).$$

Since[1]

$$\bar{\psi}(x) = (1-2\gamma_0)\psi(x), \quad \gamma_0(1-\gamma_0) = 0,$$

$$\tilde{\gamma}_\mu = \gamma_\mu, \quad \gamma_\mu(1-2\gamma_0) = -(1-2\gamma_0)\gamma_\mu,$$

(24) and (25) coincide. Equation (24) is equivalent to the second-order equations for the potentials with the only difference that in the right-hand side the Pauli current appears as well as the Dirac current. In particular, it follows from (24) that

$$\begin{aligned} \Box\psi_4(x) - \nabla_4(\nabla_\mu\psi_\mu(x)) = {} & -4\pi ie\{\bar{\varphi}(x)\beta_4\varphi(x) \\ & -l\nabla_k[\bar{\varphi}(x)\beta_4\beta_k\varphi(x)]\}. \end{aligned} \quad (27)$$

Equation (24) can be solved by utilizing the Lorentz or the Coulomb gauge for the potentials. Usually the Lorentz gauge is used for the potentials, and this causes, after second quantization, a subsidiary condition to be imposed on the state vector. The longitudinal and the scalar fields describing the Coulomb interaction are then eliminated with the aid of this condition (cf., for example, references 7 and 8). In the final result the Coulomb field remains unquantized. The use of the Coulomb gauge enables us to avoid this procedure, since the physical distinction between the transverse and the Coulomb fields is already apparent in the equations of motions themselves. Thus, the possibility appears of taking care of the Coulomb interaction in the classical theory without subjecting the field components describing this interaction to second quantization.

Such an approach has been employed by Lipmanov[9] to describe the ordinary Coulomb interaction on the basis of a relativistic generalization of the Coulomb gauge. This generalization is associated with an arbitrary choice of the space-time surface, on which all the characteristics of the system (energy, momentum etc.) are given in the Heisenberg representation. However, such a generalization of the gauge contributes nothing new to the construction of the scattering matrix. Therefore, we shall limit ourselves to taking the gauge in the form

$$\nabla_k\psi_k(x) = 0, \quad (28)$$

as a result of which the Coulomb interaction is described only by the component $\psi_4(x)$.

As will be shown later, the choice of the gauge (28) allows us to take into account the "generalized" Coulomb interaction* already in the classical theory. This leads to the result that in the interaction representation the operators of the electromagnetic field obey a first order ten-dimensional wave equation whose solutions have been given earlier.[1]

On taking (28) into account, Eq. (27) goes over into an equation for the purely Coulomb part of the potential

$$\Delta\psi_4(x) = -4\pi ie\{\bar{\varphi}(x)\beta_4\varphi(x) - l\nabla_k[\bar{\varphi}(x)\beta_4\beta_k\varphi(x)]\}. \quad (29)$$

We break up the potential $\psi_4(x)$ into two parts $\psi_4'(x)$ and $\psi_4''(x)$ which satisfy the following equations

$$\Delta\psi_4'(x) = 4\pi iel\nabla_k[\bar{\varphi}(x)\beta_4\beta_k\varphi(x)], \quad (30)$$

$$\Delta\psi_4''(x) = -4\pi ie\bar{\varphi}(x)\beta_4\varphi(x). \quad (31)$$

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*We note that $1-\gamma_0$ picks out the first four components of ψ, while γ_0 picks out the last six components. Therefore, all the terms in (22) have the same dimensions.

†We note that because of the properties[1] of the function $\psi(x)$ the following equation holds

$$\Gamma(1-\gamma_0-l\gamma_0)\psi(x) = \bar{\psi}(x)(1-\gamma_0+l\gamma_0)\Gamma;$$

so that the interaction terms in (23) are equal to $L_{\text{int}}(x)$ in (22). In carrying out the variation $\psi(x)$ and $\bar{\psi}(x)$ in (23) are regarded as independent.

*The "generalized" Coulomb interaction, just as the ordinary one, includes the Coulomb part of the interaction due to the Pauli terms.

We obtain the solution of these equations with the aid of the invariant function $\mathscr{D}(x)$ (cf., for example, reference 9). It has the form

$$\psi_4'(x) = -4\pi iel\int \frac{\partial\mathscr{D}(x-x')}{\partial x_0}\Big|_{x_0=x_0'} \nabla_k[\bar{\varphi}(x')\beta_4\beta_k\varphi(x')]\,d^3x', \tag{32}$$

$$\psi_4''(x) = 4\pi ie\int \frac{\partial\mathscr{D}(x-x')}{\partial x_0}\Big|_{x_0=x_0'} \bar{\varphi}(x')\beta_4\varphi(x')\,d^3x', \tag{33}$$

where $\mathscr{D}(x - x')$ is determined in our case by the relation

$$\mathscr{D}(x-x') = \frac{1}{(2\pi)^3}\int k_0^{-3}\sin k_0(x_0-x_0')\,e^{i\mathbf{k}(\mathbf{x}-\mathbf{x}')}\,d^3k. \tag{34}$$

We define the transverse part of the wave function $\psi(x)$ in the following manner:

$$\psi' = (0, \psi_1, \psi_2, \psi_3, \psi_{23}', \psi_{31}', \psi_{12}', \psi_{14}', \psi_{24}', \psi_{34}'). \tag{35}$$

Here $\psi_{\mu\nu}' = \psi_{\mu\nu} - \psi_{\mu\nu}''$, where by definition $\psi_{\mu\nu}''$ are expressed in terms of the Coulomb part of the potential ψ_4'': $\psi_{k4}'' = -\psi_{k4}'' = \nabla_k\psi_4''$, $\psi_{kn}'' = 0$. Then from (24) we obtain the following equation for $\psi'(x)$:

$$(\gamma_\mu\nabla_\mu + \gamma_0)\psi'(x) = 4\pi ie\bar{\varphi}(x)(1-\gamma_0+l\gamma_0)\Gamma'\varphi(x) - P(x). \tag{36}$$

Here Γ' differs from Γ by the fact that zero appears in place of β_4; $P(x)$ is a ten-dimensional vector of the following form

$$P(x) = \begin{pmatrix} 0 \\ \nabla\nabla_4\psi_4''(x) \\ 0 \\ -\nabla\psi_4'(x) \end{pmatrix}. \tag{37}$$

We have thus obtained the equation of motion for the transverse part of the electromagnetic field, with a right-hand side completely determined, in accordance with (32) and (33), by the current due to the charged Dirac particles with anomalous magnetic moment.

On going over to the quantum theory we define the canonical commutation relations for the functions $\varphi(x)$ and $\psi'(x)$ in the following manner [cf. (7) and (16)]:

$$[\beta_4\varphi(\mathbf{x}, x_0)\cdot\bar{\varphi}(\mathbf{x}', x_0)]_+ = \delta(\mathbf{x}-\mathbf{x}'), \tag{38}$$

$$[\gamma_4\psi'(\mathbf{x}, x_0)\cdot\bar{\psi}(\mathbf{x}', x_0)] = 4\pi\hbar c\hat{S}^2\gamma_4^2(1-\gamma_k\nabla_k)\,\delta(\mathbf{x}-\mathbf{x}'), \tag{39}$$

where

$$\hat{S} = (\nabla_k/\sqrt{\nabla^2})\,\delta_{klm}\gamma_l\gamma_m,$$

while the result of applying the operator $1/\sqrt{\nabla^2}$ to the function $\exp(i\mathbf{k}\cdot\mathbf{x})$ is defined by the formula

$$(1/\sqrt{\nabla^2})\,e^{i\mathbf{kx}} = e^{i\mathbf{kx}}/ik$$

(cf., for example, reference 7).

Since equations (36) contain subsidiary conditions, i.e., row-equations that do not contain the time derivative, the transition to the interaction representation cannot be carried out in the usual manner (c.f., for example, reference 10).

In order to get around this difficulty we use the following procedure. We alter the Lagrangian for the interacting fields in such a way that on introducing the interaction the subsidiary conditions remain unaltered. At the same time we require that the altered Lagrangian $L_a(x)$ should lead to the ten-dimensional equation for the electromagnetic field from which the former second-order equations for the potentials can be obtained. On taking this into account we can write

$$\begin{aligned} L_{\mathbf{a}}(x) &= (8\pi)^{-1}\bar{\psi}(x)(\gamma_\nu\nabla_\nu+\gamma_0)\psi(x) + \hbar c\bar{\varphi}(x)(\beta_\mu\nabla_\mu+\varkappa)\varphi(x) \\ &+ \tfrac{1}{2}iel\bar{\psi}(x)\;Rx + \tfrac{1}{2}ielR(x)\psi(x) - \tfrac{1}{2}ie\bar{\varphi}(x)\bar{\psi}(x)(1-\gamma_0 \\ &+ l\gamma_0)\Gamma_{\mathbf{a}}\varphi(x) - \tfrac{1}{2}ie\bar{\varphi}(x)\Gamma_{\mathbf{a}}(1-\gamma_0-l\gamma_0)\bar{\psi}(x)\varphi(x), \end{aligned} \tag{40}$$

$$\Gamma_{\mathbf{a}} = (\beta_4, \beta_1, \beta_2, \beta_3, 0, 0, 0, \beta_1\beta_4, \beta_2\beta_4, \beta_3\beta_4),$$

$$R(x) = \begin{pmatrix} 0 \\ \nabla_k\bar{\varphi}(x)\boldsymbol{\beta}\beta_k\varphi(x) \\ 0 \\ 0 \end{pmatrix}. \tag{41}$$

We note that from (40) we obtain the former equations (26) for the electron-positron field. By utilizing the previously described method of picking out the transverse part of the electromagnetic field we obtain from $L_a(x)$

$$\begin{aligned} (\gamma_\mu\nabla_\mu+\gamma_0)\psi'(x) &= 4\pi ie\bar{\varphi}(x)\Gamma_{\mathbf{a}}'(1-\gamma_0+l\gamma_0)\varphi(x) \\ &- P(x) - 4\pi ielR(x), \end{aligned} \tag{42}$$

where Γ_a' differs from Γ_a by the fact that zero appears in place of β_4. After this the transition to the interaction representation for equations (26) and (42) with the aid of the commutation relations (38) and (39) presents no difficulty. The unitary transformation $U(x_0)$ which accomplishes this transition satisfies the equation

$$i\hbar c\,dU(x_0)/dx_0 = \int H(x)\,d^3x\cdot U(x_0), \tag{43}$$

where $H(x)$ has the following form*

$$\begin{aligned} H(x) &= -ie\bar{\varphi}(x)\bar{\psi}'(x)(1-\gamma_0+l\gamma_0)\Gamma_{\mathbf{a}}'\varphi(x) + iel\bar{\psi}(x)R(x) \\ &- \tfrac{1}{2}ie\psi_4(x)\cdot\bar{\varphi}(x)\beta_4\varphi(x) \\ &+ \tfrac{1}{2}iel\nabla_k\psi_4(x)\cdot\bar{\varphi}(x)\,\beta_k\beta_4\varphi(x). \end{aligned} \tag{44}$$

Since all the operators in (44) satisfy the free field equations:

$$\begin{aligned} (\gamma_\mu\nabla_\mu+\gamma_0)\psi'(x) = 0, &\qquad (\beta_\mu\nabla_\mu+\varkappa)\varphi(x) = 0, \\ (-\gamma_\mu\nabla_\mu+\gamma_0)\bar{\psi}'(x) = 0, &\qquad (\tilde{\beta}_\mu\nabla_\mu-\varkappa)\bar{\varphi}(x) = 0, \end{aligned} \tag{45}$$

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*All the terms in H(x) are normal products of operators taken in the interaction representation.

we can write H(x) in the following form

$$H(x) = -ie\bar{\varphi}(x)\hat{\psi}'(x)\varphi(x) - \tfrac{1}{2}ie\psi_4(x)\cdot\bar{\varphi}(x)\beta_4\varphi(x)$$
$$+\tfrac{1}{2}iel\nabla_k\psi_4(x)\cdot\bar{\varphi}(x)\beta_k\beta_4\varphi(x) + iel\nabla_n(\psi_k(x)\cdot\bar{\varphi}(x)\beta_k\beta_n\varphi(x)),$$

where the following notation has been introduced

$$\hat{\psi}'(x) = \Gamma(1-\gamma_0 - l\gamma_0)\psi'(x). \qquad (46)$$

When H(x) is integrated over all space the term $iel\nabla_n[\psi_k(x)\cdot\bar{\varphi}(x)\beta_k\beta_n\varphi(x)]$ disappears, and in constructing the scattering matrix we can utilize for H(x) the following expression

$$H(x) = H_{\mathrm{ph}}(x) + H_{\mathrm{C}}(x), \qquad (47)$$

where

$$H_{\mathrm{ph}}(x) = -ie\bar{\varphi}(x)\hat{\psi}'(x)\varphi(x), \qquad (48)$$

$$H_{\mathrm{C}}(x) = -\tfrac{1}{2}ie\psi_4(x)\cdot\bar{\varphi}(x)\beta_4\varphi(x)$$
$$+\tfrac{1}{2}iel\nabla_k\psi_4(x)\cdot\bar{\varphi}(x)\beta_k\beta_4\varphi(x). \qquad (49)$$

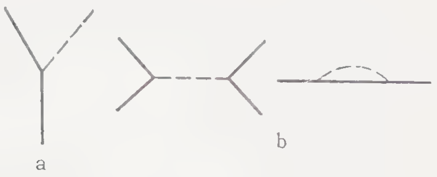
By utilizing (43) we find the following expression for the S matrix:

$$S = \sum_{n=0}^{\infty}\frac{(-i)^n}{(\hbar c)^n n!}\int dx_1 \ldots \int dx_n T\{H(x_1)H(x_2)\ldots H(x_n)\}. \qquad (50)$$

Thus, with the aid of the foregoing procedure the S matrix is finally expressed in terms of quantities referring to the Lagrangian (23).

The scattering matrix in the form (50) is not convenient for practical use since H(x) consists of two terms of the first and the second degree in terms of the Sommerfeld fine structure constant. However, it can be written in the form of a series in powers of $e/\sqrt{\hbar c}$ by a method similar to that proposed by Lipmanov.[9] It turns out to be possible formally to introduce symmetrical pairings of photon operators, with the state function, as before, not containing scalar photon states. Just as in the case of the usual scattering matrix, it is convenient in our case to use a graphical representation of the matrix elements. The part of the interaction operator $H_{\mathrm{ph}}(x)$ is represented by the simplest graph (cf., the figure, graph a) with the only difference that in the momentum representation the end of the photon line corresponds, as will be seen from subsequent discussion, to the matrix $\gamma_\mu + \tfrac{1}{2}il(\gamma_\mu\hat{k} - \hat{k}\gamma_\mu)$ instead of to γ_μ. It can be easily shown that the graph corresponding to the Coulomb part of the interaction operator $H_{\mathrm{C}}(x)$ must contain four electron lines, while the photon line must connect two vertices (graph b in the figure).

Therefore, it is convenient to deal with the Coulomb parts of the matrix element by considering the sum of these parts and the corresponding parts arising from the "pairing" of the photon operators $\psi'(x)$. The summation of the pairings of the photon operators with the corresponding Coulomb parts is similar to that carried out in reference 9. We shall carry this out for the sum of the second order terms of the S matrix:



$$S^{(2)}_{\mathrm{pair}} = \frac{(-i)^2}{2!(\hbar c)^2}\int dx_1 dx_2 T\{[-ie\bar{\varphi}(x_1)\hat{\psi}'\underbrace{(x_1)\varphi(x_1)][-ie\bar{\varphi}(x_2)}\hat{\psi}'$$
$$\times(x_2)\varphi(x_2)]\} - \frac{i}{\hbar c}\int dx_0\left\{-\frac{ie}{2}\int\psi_4(x)\cdot\bar{\varphi}(x)\beta_4\varphi(x)d^3x\right.$$
$$\left.+\frac{iel}{2}\int\nabla_m\psi_4(x)\cdot\bar{\varphi}(x)\beta_m\beta_4\varphi(x)d^3x\right\}. \qquad (51)$$

Since for the solution of practical problems we are interested in pairings of operators in convolution with the matrix vectors Γ, it appears advantageous to take this fact into account from the outset. Let us evaluate an expression of the following form

$$F'(x-x') = \underbrace{\hat{\psi}'(x)\ldots\hat{\psi}'}(x') = T[\hat{\psi}'(x)\ldots\hat{\psi}'(x')]$$
$$-N[\hat{\psi}'(x)\ldots\hat{\psi}'(x')], \qquad (52)$$

where the dots denote intermediate terms appearing in the scattering matrix, while the symbols T and N refer only to the operators appearing in the expressions $\psi'(x)$. By utilizing the method employed in reference 3, and on substituting (2) into (52), we obtain

$$F'(x-x') = \frac{2\hbar c}{(2\pi)^2}\int\sum_{q=\pm 1}(\hat{\psi}^{(q)}(k)\ldots\hat{\psi}^{(q)}$$
$$\times(-k))e^{-ik_0|x_0-x_0'|}e^{i\mathbf{k}(\mathbf{x}-\mathbf{x}')}\frac{d^3k}{k_0}. \qquad (53)$$

Let us transform the quantities $\hat{\psi}^{(\pm 1)}(\pm\mathrm{k})$. For example, on taking (4) into account we obtain

$$\hat{\psi}^{(+1)}(k) = \hat{\mathbf{e}} + il[e_1\beta_1(k_2\beta_2 + k_3\beta_3 + k_4\beta_4)$$
$$+ e_2\beta_2(k_1\beta_1 + k_3\beta_3 + k_4\beta_4) + e_3\beta_3(k_1\beta_1 + k_2\beta_2 + k_4\beta_4)], \qquad (54)$$

where the indices 1 — 4 denote the components of the vectors **e** and **k**; $\hat{\mathbf{e}} = e_a\beta_a$. On adding to (54) a sum of terms of the form

$$il(e_1\beta_1\cdot k_1\beta_1 + e_2\beta_2\cdot k_2\beta_2 + e_3\beta_3\cdot k_3\beta_3) = il\mathbf{ek} = 0,$$

we obtain the expression

$$\hat{\psi}^{(+1)}(k) = \hat{\mathbf{e}} + \tfrac{1}{2}il(\hat{\mathbf{e}}\hat{k} - \hat{k}\hat{\mathbf{e}}),$$

where $\hat{k} = k_\mu \beta_\mu$. In a similar manner we obtain for the remaining expressions $\hat{\psi}^{(\pm 1)}(\pm k)$ the relations

$$\hat{\psi}^{(\pm 1)}(\pm k) = \hat{e} \pm \tfrac{1}{2} il(\hat{e}\hat{k} - \hat{k}\hat{e}),$$
$$\hat{\psi}^{(\mp 1)}(\pm k) = \hat{e}^* \pm \tfrac{1}{2} il(\hat{e}^*\hat{k} - \hat{k}\hat{e}^*). \tag{55}$$

Similar relations also hold for $\hat{\psi}^{(0)}(\pm k)$, where the vectors $\psi^{(0)}(\pm k)$ describe the free field longitudinal photon states:

$$\hat{\psi}^{(0)}(\pm k) = \hat{k} = \hat{k} \pm \tfrac{1}{2} il(\hat{k}\hat{k} - \hat{k}\hat{k}). \tag{56}$$

On substituting (55) into (53) and on taking into account the fact[1] that $e \cdot e^* + e^* \cdot e = 1 - k \cdot k/k^2$, we can write

$$F'(x - x') = 4\pi\hbar c\left\{\left[\beta_a + \frac{l}{2}(\beta\,\hat{\nabla} - \hat{\nabla}\beta_a)\right]\dots\right.$$
$$\left.\dots\left[\beta_b - \frac{l}{2}(\beta_b\hat{\nabla} - \hat{\nabla}\beta_b)\right]\right\}\left(1 - \frac{\nabla \cdot \nabla}{\nabla^2}\right)_{ab} D^F(x - x'). \tag{57}$$

Here $\hat{\nabla}$ denotes $\beta_\mu \nabla_\mu$, while $D^F(x - x')$ is the well-known singular Feynman function.

We write expression (57) as a sum of two terms:

$$F'(x-x') = 4\pi\hbar c\{[\beta_\mu + \tfrac{1}{2} l(\beta_\mu\hat{\nabla}\ \ \hat{\nabla}\beta_\mu)]\dots$$
$$\times[\beta_\mu - \tfrac{1}{2} l(\beta_\mu\hat{\nabla} - \hat{\nabla}\beta_\mu)]\} D^F(x-x')$$
$$- 4\pi\hbar c\{[\beta_4 + \tfrac{1}{2} l(\beta_4\hat{\nabla} - \hat{\nabla}\beta_4)]\dots[\beta_4 - \tfrac{1}{2} l(\beta_4\hat{\nabla} - \hat{\nabla}\beta_4)]$$
$$+ [\beta_a + \tfrac{1}{2} l(\beta_a\hat{\nabla} - \hat{\nabla}\beta_a)]\dots$$
$$\times[\beta_b + \tfrac{1}{2} l(\beta_b\hat{\nabla} - \hat{\nabla}\beta_b)]\,\nabla_a\nabla_b/\nabla^2\} D^F(x-x'). \tag{58}$$

After some transformations, the first term of the sum can be rewritten in the form

$$F(x - x') = \frac{4\pi\hbar c}{(2\pi)^4 i}\int\frac{d^4k}{k^2 - i\varepsilon}\left[\beta_\mu + \frac{il}{2}(\beta_\mu\hat{k} - \hat{k}\beta_\mu)\right]\dots$$
$$\times\left[\beta_\mu - \frac{il}{2}(\beta_\mu\hat{k} - \hat{k}\beta_\mu)\right]e^{ik(x-x')}. \tag{59}$$

By utilizing the equation of continuity for the current density we can show that the substitution of the second term of (58) into (51) leads to an expression that cancels the second term of (51).

Thus, if we formally define the pairing of the photon operators in convolution with the matrix vectors Γ by equation (59), then the expression for $S^{(2)}_{\text{pair}}$ may be written in the form

$$S^{(2)}_{\text{pair}} = \frac{1}{2!}\left(-\frac{e}{\hbar c}\right)^2\int dx\,dx'\,T\{\bar{\varphi}(x)\underline{\hat{\psi}'(x)\varphi(x)\cdot\bar{\varphi}(x')\hat{\psi}'}(x')\varphi(x'), \tag{60}$$

where in contrast to formula (51) we have adopted a different notation for the pairing since it is defined in a different manner. From a comparison of (51) and (60) it follows that

$$\frac{e^2}{\hbar c}\int dx\,dx'\,T[\bar{\varphi}(x)\hat{\psi}'\underbrace{(x)\varphi(x)\cdot\bar{\varphi}(x')}\hat{\psi}'(x')\varphi(x')]$$
$$= \frac{e^2}{\hbar c}\int dx\,dx'\,T[\bar{\varphi}(x)\underline{\hat{\psi}'(x)\varphi(x)\cdot\bar{\varphi}(x')\hat{\psi}'}(x')\varphi(x')]$$
$$+ 2i\int H_\kappa(x)\,dx. \tag{61}$$

It may be easily shown that (61) holds also in the case when the left-hand side appears as an integral factor in the higher order terms of the S matrix.

This enables us to draw a more general conclusion, viz., that the complete scattering matrix (50) is equivalent to the following expansion:*

$$S \sim \sum_{n-0}^{\infty}\left(-\frac{e}{\hbar c}\right)^n\frac{1}{n!}\int dx_1\dots\int dx_n\,T[\bar{\varphi}(x_1)\hat{\psi}'(x_1)$$
$$\times\varphi(x_1)\dots\bar{\varphi}(x_n)\hat{\psi}'(x_n)\varphi(x_n)]. \tag{62}$$

The equivalence should be interpreted in the sense that the matrix elements of S are obtained from the matrix elements of the right-hand side of (62) if all the pairings of the photon operators combined with Γ are replaced in them by expressions (59).

For the solution of problems involving the electromagnetic interaction of two different Dirac particles the interaction Lagrangian is constructed from two terms of type (22). In this case the scattering matrix (62) assumes the following form

$$S \sim \sum_{n-0}^{\infty}\left(-\frac{e}{\hbar c}\right)^n\frac{1}{n!}\int dx_1\dots\int dx_n\,T\{[\bar{\varphi}_1(x_1)\hat{\psi}'_1(x_1)\varphi_1(x_1)$$
$$+\bar{\varphi}_2(x_1)\hat{\psi}'_2(x_1)\varphi_2(x_1)]\dots$$
$$\dots[\bar{\varphi}_1(x_n)\hat{\psi}'_1(x_n)\varphi_1(x_n) + \bar{\varphi}_2(x_n)\hat{\psi}'_2(x_n)\varphi_2(x_n)]\}, \tag{63}$$

where the indices 1 and 2 carried by φ and $\bar{\varphi}$ indicate different types of Dirac fields, while the indices 1 and 2 carried by $\hat{\psi}'$ indicate different Pauli constants l_1 and l_2.

The application of the matrix (62) to the calculations of the Compton scattering by a proton, and correspondingly, the application of (63) to the calculation of the Møller scattering of two different Dirac particles leads to the same results as Feynman's method.

In the scattering matrix (62) only the transverse photon operators appear. In reference 1 in addition to the transverse photon operators the free field longitudinal photon operators were also formally introduced. The longitudinal photon states need not have been taken into account, since the energy of these states is equal to zero. However, in the case when effects involving interaction with unpolarized light are being discussed all the calculations can be considerably simplified with their aid. It may be easily shown that just as in the case of the usual scattering matrix (cf., for example, reference 4) the $\psi^{(0)}$ states do not give rise to any effects in real processes. In the momentum representation the matrix element which corresponds to a transition from the state in which the electron has momentum p_0 to the state in which it has momentum $p_N + k$, is of the form

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*The proof of this is completely analogous to the one given in reference 9, and is therefore not given here.

$$M_q = \varphi(p_N + k)\Big\{\hat{\psi}_N \prod_{j=1}^{N-1} A_j\hat{\psi}_j A_0\hat{\psi}^{(q)}(k)$$

$$+ \sum_{s=1}^{N-1} \hat{\psi}_N \prod_{j=s+1}^{N-1} A_j\hat{\psi}_j A_s\hat{\psi}^{(q)}(k) \prod_{j=1}^{s} B_j\hat{\psi}_i$$

$$+ \hat{\psi}^{(q)}(k) \prod_{j=1}^{N} B_j\hat{\psi}_i\Big\} \varphi(p_0). \qquad (64)$$

Here the quantities $\psi_j = \psi^{(q_j)}(k_j)$ are the Fourier components of the electromagnetic field functions which represent the corresponding perturbing fields

$$A_j = [\varkappa - i(\hat{p}_j + \hat{k})] / [\varkappa^2 + (p_j + k)^2],$$

$$B_j = (\varkappa - i\hat{p}_j) / (\varkappa^2 + p_j^2), \quad p_j = p_{j-1} + k_j,$$

and in the products Π the factors are written from left to right in the order of decreasing j. In the matrix element the Fourier component of the perturbing field $\psi^{(q)}(k)$ is specifically picked out. The terms of the sum in (64) correspond to graphs which differ from one another by the position of the photon line representing $\psi^{(q)}(k)$.

Let $\psi^{(q)}(k)$ correspond to the emission or absorption of a longitudinal free field photon, i.e., $q = 0$. Proceeding in a manner analogous to that employed by Feynman,[4] it can then be easily shown that

$$M_0 = 0. \qquad (65)$$

This enables us to simplify considerably the calculation of the transition probabilities for unpolarized light. In this case the quantity $|M_{+1}|^2 + |M_{-1}|$ is to be evaluated. We introduce the notation

$$\mathfrak{M}_\mu = \bar{\varphi}(p_N + k)\Big\{\hat{\psi}_N \prod_{j=1}^{N-1} A_j\hat{\psi}_j A_0\beta_\mu(1 + il\hat{k})$$

$$+ \sum_{s=1}^{N-1} \hat{\psi}_N \prod_{j=s+1}^{N-1} A_j\hat{\psi}_j A_s\beta_\mu(1 + il\hat{k}) \prod_{j=1}^{N} B_j\hat{\psi}_j$$

$$+ \beta_\mu(1 + il\hat{k}) \prod_{j=1}^{s} B_j\hat{\psi}_j\Big\} \varphi(p_0). \qquad (66)$$

By utilizing the dyadic notation and by taking (55) into account, we obtain

$$|M_{+1}|^2 + |M_{-1}|^2 = \mathfrak{M}_a\mathfrak{M}_b^*(\mathbf{e}\cdot\mathbf{e}^* + \mathbf{e}^*\cdot\mathbf{e})_{ab}. \qquad (67)$$

On adding to (67) a term of the form

$$|\mathbf{k}|^{-2}[\mathfrak{M}_a\mathfrak{M}_b^*(\mathbf{k}\cdot\mathbf{k})_{ab} - \mathfrak{M}_4\mathfrak{M}_4^* k_4^2],$$

which in consequence of (65) and (56) is equal to zero, we obtain

$$|M_{+1}|^2 + |M_{-1}|^2 = \mathfrak{M}_a\mathfrak{M}_b^*(\mathbf{e}\cdot\mathbf{e}^* + \mathbf{e}^*\cdot\mathbf{e} + \mathbf{k}\cdot\mathbf{k}/\mathbf{k}^2)_{ab}$$

$$+ \mathfrak{M}_4\mathfrak{M}_4^* = \mathfrak{M}_\mu\mathfrak{M}_\mu^*. \qquad (68)$$

Thus, the method of simplifying the calculation of the transition probabilities for unpolarized light proposed by Feynman[4] is generalized in a natural manner also to the Pauli interaction. In subsequent papers the method just presented will be applied to the calculation of different effects due to interactions.

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SOME PROPERTIES OF STATIONARY FLOWS IN MAGNETIC GAS DYNAMICS

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Stationary quasi-one-dimensional and one-dimensional flows are considered. It is shown that shock wave formation is possible under certain conditions in the case of media with finite conductivity.

STEADY one-dimensional and quasi-one-dimensional flows (by the latter are meant flows with a variable, smoothly changing cross section $f(x)$ filled by the given lines of flow) of an arbitrarily conducting medium, assumed here to be an ideal gas, are described for the case of a perpendicular magnetic field $[\mathbf{u} \perp \mathbf{H},\ \mathbf{u}(u, 0, 0),\ \mathbf{H}(0, h\sqrt{\pi}, 0),\ \mathbf{E}(0, 0, e\sqrt{\pi})]$ by a set of stationary equations of magnetic gas dynamics, two of which have as first integrals[1]

$$d(\rho u f) / dx = 0, \qquad \rho u f = M, \tag{1}$$

$$\rho u\, du / dx + dp_m / dx = 0, \tag{2}$$

$$d(hu) / dx = \nu_m d^2 h / dx^2, \qquad hu = \nu_m dh/dx + MB, \tag{3}$$

$$u dp - u a^2 d\rho = \nu_m (k - 1)(dh)^2 / dx \tag{4}$$

(u, p, ρ, H, E are the velocity, pressure, density, magnetic and electric field, respectively; M and B are constants; $p_m = p + h^2/2$; ν_m is the "magnetic" viscosity, k is the ratio of specific heats, and $a = (kp/\rho)^{1/2}$ is the speed of sound).

Let us consider the stationary quasi-one-dimensional flows of media with $\nu_m = 0$. We have from the system (1) — (4)

$$d\rho / du = -(\rho / u a^2)(u^2 - h^2 / \rho), \tag{5}$$

$$d(\rho u) / du = \rho [1 - a^{-2}(u^2 - h^2 / \rho)]. \tag{6}$$

It is not difficult to prove that the current density ρu achieves its maximum value at the critical point for which $u = a_{m0} = (a^2 + h^2/\rho)^{1/2}$. The following relations can also be obtained from Eqs. (1)—(4):

$$(u^2 - a_{m0}^2)\, du / u = a^2 df / f, \tag{7}$$

$$(u^2 - a_{m0}^2)\, d\rho / \rho = -(u^2 - h^2 / \rho)\, df / f, \tag{8}$$

whence it follows that for $u < a_{m0}$, if $df \gtrless 0$, then $du \lessgtr 0$; for $u > a_{m0}$, if $df \gtrless 0$, then $du \lessgtr 0$. It is interesting that the extremum of ρ takes place for $u^2 = h^2/\rho$, which is seen from (5) ($h/\sqrt{\rho}$ is the velocity of propagation of Alfven waves). It follows from (8) that for $df < 0$, if $u^2 \lessgtr h^2/\rho$, then $d\rho \gtrless 0$; for $df > 0$, if $u^2 \lessgtr h^2/\rho$, then $d\rho \lessgtr 0$, i.e., in a flow with converging flow lines we have a maximum ρ for $u^2 = h^2/\rho$, and with diverging lines we have a minimum ρ.

We consider stationary one-dimensional flows of media with $\nu_m \neq 0$. If $f = \text{const}$, then it is necessary to replace the input M and the constant B in the set (1) — (4) by the current density $m = M/f$ and the constant $b = Bf$, which is expressed in terms of the constant value of the electric field $e = e_0$: $b = -ce_0/m$. Here Eq. (2) has the integral

$$mu + p_m = I. \tag{2a}$$

We further transform Eq. (4) by means of (1) and (3) in such a fashion that the differential dp_m appears in place of dp. We obtain an equation similar to Eq. (4):

$$u dp_m - u a_m^2 d\rho = \nu_m (k - 1)(dh / dx)^2 dx, \tag{4a}$$

where

$$a_m = \left(a^2 + h \frac{dh}{d\rho}\right)^{1/2} = (a^2 + \Delta_1 + \Delta_2)^{1/2}$$

$$= \left(a^2 + \frac{h^2}{\rho} - \nu_m \frac{h}{\rho} \frac{d^2h / dx^2}{du/dx}\right)^{1/2} \tag{9}$$

coincides with the expression for the propagation velocity of low-intensity shock waves in an arbitrarily conducting medium in the presence of a perpendicular magnetic field.[1] We can also verify that a_m has the meaning of an effective sound velocity by linearizing the nonstationary system corresponding to Eqs. (1) — (3), (4a) in the usual way. When $\nu_m = 0$ we have $h = \text{const} \cdot \rho$ and $a_m = a_{m0}$; for $\nu_m \neq 0$ the value of a_m depends essentially on the flow conditions. It is evident that if Δ_1 is due to the magnetic field, then the presence of Δ_2 can be explained by the finite conductivity; in this case it is natural to assume[1] that

$$a_{m0} \geqslant a_m \geqslant a. \tag{10}$$

The "freezing-in" condition follows from (1) and (3):

$$h / \rho = (\nu_m / m)\, dh / dx + b, \tag{11}$$

while the equation

$$(u^2 - a_m^2)\, d\rho / \rho = \nu_m (k-1)\, m^{-1} (dh/dx)^2 dx = (k-1)\, dQ, \tag{12}$$

can be obtained from (1), (2a) and (4a), where

$$dQ = (\nu_m / m)(dh / dx)^2\, dx = T dS \tag{13}$$

is the Joule heat. If the components Δ_1 and Δ_2, which are obtained in terms of a_m from (9), are transferred to the right-hand side in (12), then, by taking (11) into account, we have

$$(u^2 - a^2)\, d\rho / \rho = (k \nu_m m^{-1} dh / dx + b)\, dh. \tag{14}$$

With the help of the well-known thermodynamic relations that hold for an ideal gas

$$dw = dQ + dp / \rho = k dQ + a^2 d\rho / \rho \tag{15}$$

(w is the enthalpy), and making use of (13) and (1), we reduce Eq. (14) to an integrable form

$$\begin{gathered} u du + dw + b dh = 0, \\ u^2 / 2 + w + bh = u^2 / 2 + w_m = A = \text{const}. \end{gathered} \tag{16}$$

It is easy to prove that w_m has the meaning of effective enthalpy only when $\nu_m = 0$.

We can use the integral (16) in place of (4a); thus, for example, we can obtain from (1), (2a), (3), and (4a) a single equation for dh/dx, which can be integrated numerically.[1]

We note that Eq. (12) recalls the well-known equation of classical gas dynamics for the motion of a viscous gas in a heat-insulated pipe of constant cross section.[2] Since we shall not consider the effect of thermal conduction, and since the motions of media with finite conductivity are accompanied by dissipation and the production of heat, such an analogy becomes obvious.

The right side in (12) can only be positive; therefore,

$$\begin{gathered} d\rho < 0, \quad du > 0 \text{ for } u < a_m; \\ d\rho > 0, \quad du < 0 \text{ for } u > a_m. \end{gathered} \tag{17}$$

Continuous transition through the critical velocity $u = a_m$ is impossible (flow crisis). For $u = a_m$, at the point $x = x_k$, we have

$$dh / dx\,\big|_{x_k} = 0 \tag{18}$$

and, as follows from (9),

$$a_m (x_k) = a (x_k) = u (x_k). \tag{19}$$

The values of u, ρ, p, h at the point x_k are completely determined from (1), (2a), (3), and (4a) in terms of the constants m, b, J, and A (we also have $a_m = a$ at the point in the current where $dh/dx = -mb/\nu_m$ and $h = 0$). The relations (18) and (19) satisfy Eq. (14), which gives in addition another possible value of dh/dx for the flow velocity $u = a\ (x = x^*)$:

$$dh / dx\,|_{x^*} = - mb / k\nu_m. \tag{20}$$

If we take into account the entropy S as a function of ρ, then it is seen from (12) that when $u = a_m$ this function has an extremum which can only be a maximum, since there is no $S = S(\rho)$ that has a minimum when $u = a_m$ and satisfies simultaneously (17) and the condition $dS > 0$. In this connection, we can conclude the following: if at any point in the current $x = x_0$, taken as the initial point, we have $u > a_m$ while $S = S_{max}$ is attained at a finite distance from x_0, then the formation of a shock wave is inevitable, behind which $u < a_m$; if $S = S_{max}$ is attained at infinity or is not attained at all, then $u > a_m$ at any finite distance from x_0; if at the point x_0 we have $u > a_m$, then the value $u = a_m$ is a maximum and is attained only at infinity. However, it is necessary to recall that in media with high conductance (small value of ν_m) there is a high thermal conductivity, an account of which can lead to qualitatively different results.

We now determine the possible values of dh/dx in the current, at the initial point of which $u(x_0)$ and $h(x_0) > 0$, while $e = e_0 < 0$. It is obvious from (11) that $dh/dx \geq -mb/\nu_m$. It follows from (14) that for $u < a$,

$$\begin{aligned} dh / dx > 0 \text{ or } - mb / \nu_m \leqslant dh / dx < - mb / k\nu_m, &\quad du > 0, \\ - mb / k\nu_m < dh / dx < 0, &\quad du < 0; \end{aligned} \tag{21a}$$

for $u > a$,

$$\begin{aligned} - mb / k\nu_m < dh / dx < 0, &\quad du > 0, \\ dh / dx > 0 \text{ or } - mb / \nu_m \leqslant dh / dx < - mb / k\nu_m, &\quad du < 0. \end{aligned} \tag{21b}$$

Comparing (21), (17), (9), and (10) we get

for $u < a$, a_m ($du > 0$, $d\rho < 0$):

$$- mb / \nu_m \leqslant dh / dx < - mb / k\nu_m, \qquad d^2 h / dx^2 > 0; \tag{22a}$$

for $a < u < a_m$ ($du > 0$, $d\rho < 0$):

$$- mb / k\nu_m < dh/dx < 0, \qquad d^2 h / dx^2 > 0; \tag{22b}$$

for $u > a$, a_m ($du < 0$, $d\rho > 0$).

$$dh / dx > 0, \quad d^2 h / dx^2 < 0. \tag{22c}$$

It is clear from (22) that the value of dh/dx

$= -mb/\nu_m$ $(h = 0)$ is impossible while, generally speaking, a point $x = x^*$ can exist in a current with $u < a_m$.

The values of dh/dx can be similarly estimated for other flow conditions.

In conclusion, the author expresses his gratitude to Prof. K. P. Stanyukovich and G. S. Golitsyn for a number of discussions.

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SOVIET PHYSICS JETP VOLUME 12, NUMBER 2 FEBRUARY, 1961

EXCITATION SPECTRUM OF A SYSTEM OF ELECTRONS AND IONS SITUATED IN A HOMOGENEOUS MAGNETIC FIELD

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The effect of a magnetic field on the spectrum of the acoustic branch of excitations of a system of electrons and ions is studied with the aid of the quantum kinetic equation.

At the present time the developed mathematical methods[1-4] of taking into account the collective Coulomb interactions in a system of many particles are being used more and more widely in solid state theory. In this connection the solid (metal) is approximated by an isotropic plasma and in the first approximation the periodic distribution of ions situated at the lattice points of a crystalline lattice is neglected. However, in order to explain those physical properties of such a plasma which are temperature-dependent (for example, electrical conductivity), the motion of the ions should be taken into account.

Silin[5,6] has studied the weakly excited states of electrons and ions by means of a quantum kinetic equation with a self-consistent field. By utilizing the quantum distribution function in the mixed representation, he has established the limits of applicability of the results obtained by means of the classical equation with a self-consistent field.[1] The analysis of the dispersion equation led to the conclusion that in the long-wave domain the excitations are of an essentially collective character.

In the present note we give the results of an investigation by the same method of the effect of a constant homogeneous magnetic field on the spectrum of the longitudinal excitations of the system under discussion. In this discussion our attention is concentrated only on the long-wave part of the spectrum which plays the fundamental role in many physical phenomena.

The dispersion equation can be easily obtained by setting up the equations for the distribution functions for the electrons $f_e(\mathbf{p}, \mathbf{q})$ and for the ions $f_i(\mathbf{p}, \mathbf{Q})$ ($\mathbf{q}$, $\mathbf{Q}$ are the coordinates and $\mathbf{p}$, $\mathbf{P}$ are the momenta of the particles). For weakly excited states we set

$$f_e(\mathbf{q},\ \mathbf{p}) = f_{0e}(p) + \varphi(\mathbf{q},\ \mathbf{p},\ t),$$
$$f_i(\mathbf{Q},\ \mathbf{P}) = f_{0i}(P) + \Phi(\mathbf{Q},\ \mathbf{P},\ t),$$

where we consider that the increments φ and Φ of the equilibrium functions are small.

By going over then to the Fourier components φ_k, Φ_k, and by solving the system of equations determining them, we obtain the spectrum of the collective oscillations. In doing this we follow Landau's method[7] (we carry out a Laplace-Mellin transformation with respect to time). The dispersion equation will then have the form

$$\begin{vmatrix} 1-\nu_{ee}(k)\,a_e, & -\nu_{ei}(k)\,a_e \\ -\nu_{ie}(k)\,a_i, & 1-\nu_{ii}(k)\,a_i \end{vmatrix} = 0, \tag{1}$$

where

$$a_j = -\frac{i}{\Omega_j\hbar}\int \exp\left\{\frac{1}{\Omega_j}\left[\left(s+i\frac{k_zp_z}{m_j}\right)\vartheta+i\frac{k_xp_r}{m_j}\sin\vartheta\right]\right\}p_r\,dp_r\,dp_z\,d\vartheta$$
$$\times\left[\int_0^{\vartheta}\left[f_{0j}\left(\left|\mathbf{p}+\frac{\hbar\mathbf{k}}{2}\right|\right)-f_{0j}\left(\left|\mathbf{p}-\frac{\hbar\mathbf{k}}{2}\right|\right)\right]\right.$$
$$\times\exp\left\{-\frac{1}{\Omega_j}\left[\left(s+i\frac{k_zp_z}{m_j}\right)\beta+i\frac{k_xp_r}{m_j}\sin\beta\right]\right\}d\beta$$
$$+\frac{\exp\{(s+ik_zp_z/m_j)\,2\pi/\Omega_j\}}{1-\exp\{(s+ik_zp_z/m_j)\,2\pi/\Omega_j\}}\int_0^{2\pi}\left[f_{0j}\left(\left|\mathbf{p}+\frac{\hbar\mathbf{k}}{2}\right|\right)\right.$$
$$\left.-f_{0j}\left(\left|\mathbf{p}-\frac{\hbar\mathbf{k}}{2}\right|\right)\right]\times\exp\left\{-\frac{1}{\Omega_j}\left[\left(s+i\frac{k_zp_z}{m_j}\right)\beta\right.\right.$$
$$\left.\left.\left.+i\frac{k_xp_r}{m_j}\sin\beta\right]\right\}d\beta\right];$$

$$j=e,\ i;\qquad \Omega_j=e_jH/m_jc;\qquad m_e=m,\qquad m_i=M;$$

s is the Laplace-Mellin expansion parameter; the z axis coincides with the direction of the magnetic field $\mathbf{H}$; the angle ϑ is measured from the $\mathbf{H}$, $\mathbf{k}$ plane; $\nu_{ij}(\mathrm{k}) = \int U_{ij}(\mathrm{r})\exp(i\mathbf{kr})\,d\tau$ are the Fourier components of the potential energies of interaction $U_{ij}(\mathrm{r})$ between corresponding particles.

In the case of the Coulomb interaction $\nu_{ee}(k) = 4\pi e_e^2/k^2$; $\nu_{ii}(k) = 4\pi e_i^2/k^2$; $\nu_{ei}\nu_{ie} = \nu_{ee}\nu_{ii}$ and expression (1) reduces to the form

$$1-\nu_{ee}(k)\,a_e-\nu_{ii}(k)\,a_i = 0.$$

The long waves correspond to small values of $\mathbf{k}$ so that in the integral we can set

$$\frac{1}{\hbar}\left(f_0\left(\left|\mathbf{p}+\frac{\hbar\mathbf{k}}{2}\right|\right)-f_0\left(\left|\mathbf{p}-\frac{\hbar\mathbf{k}}{2}\right|\right)\right)=\mathbf{k}\frac{\partial f_0}{\partial\mathbf{p}} \qquad (2)$$

and, moreover,

$$\begin{aligned}&\hbar^2k^2/2m\ll n_{0e}\nu_{ee}(k), && p_e^2/2m\gg\hbar^2k^2/2m;\\ &\hbar^2k^2/2M\ll n_{0i}\nu_{ii}(k), && p_i^2/2M\gg\hbar^2k^2/2M;\end{aligned} \qquad (2a)$$

n_{0i}, n_{0e} are the equilibrium concentrations of the ions and of the electrons.

Relation (2) does not exclude the possibility that the distribution in the case of the equilibrium functions is of a quantum nature. In the absence of a magnetic field expression (1) gives for small k two branches of oscillations, acoustic and electronic.[5,6] Thermal motion in a metal excites only the acoustic branch. As regards the electron oscillations, temperatures of the order of 10^4°K are needed to excite them.

We assume that the temperature satisfies $T > \Theta$ where Θ is the Debye temperature. In practice, the electron gas in a metal is degenerate at all temperatures, so that we use the Fermi distribution function for f_{0e}. For the ions we take the Maxwellian function.

In order to take into account the effect of the magnetic field on the acoustic branch we set

$$p_{0e}^2/m^2\gg\omega^2/k^2\gg p_i^2/M^2, \qquad (3)$$

where p_{0e} is the limiting momentum of the Fermi distribution corresponding to the energy $\epsilon_0 = p_{0e}^2/2m$. We neglect the damping of the waves, and on setting $s = -i\omega$ we obtain

$$a_i=-2\left(\frac{M}{2\pi\varkappa T}\right)^{3/2}\frac{n\pi_{0i}}{\varkappa T}\left\{\frac{(2\pi)^{1/2}(\varkappa T)^{3/2}}{M}\right.$$
$$\left.-\omega\sum\int_{-\infty}^{\infty}\frac{\exp\{-Mv_z^2/2\varkappa T\}\,dv_z}{\omega-k_zv_z-n\Omega_i}\int_0^{\infty}J_n^2\left(\frac{k_xv_r}{\Omega_i}\right)e^{-Mv_r^2/2\varkappa T}v_r\,dv_r\right\}.$$

On taking (3) into account, and on using the expansion for the Bessel function, we obtain approximately

$$a_i=\frac{n_{0i}}{M}\cdot\left(\frac{k_z^2}{\omega^2}-\frac{k_x^2}{\omega^2-\Omega_i^2}\right).$$

a_e is evaluated in a somewhat different manner. Then on substituting the resultant values of a_i and a_e into (1) we obtain

$$\omega^2=\frac{1}{2\beta}\{c^2k^2+\beta\Omega_i^2\pm[(c^2k^2+\beta\Omega_i^2)^2-4\beta c^2\Omega_i^2k^2\cos^2\alpha]^{1/2}\}, \qquad (4)$$

where

$$\beta=1+(\pi^2/24)(\varkappa T/\varepsilon_0)^2, \qquad c^2=(p_{0e}^2/3Mm)|e_i/e_e|.$$

Expression (4) taken with a minus sign defines the acoustic branch; for small values of k (4) yields

$$\omega=(ck/\sqrt{\beta})\cos\alpha, \qquad (5)$$

α is the angle between **H** and **k**.

For $T = 0$ and $\cos\alpha = 1$ (the magnetic field is absent) we recover Silin's result.[6]

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DYNAMIC STABILIZATION OF A PLASMA RING

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A method is proposed for dynamic stabilization of long-wave instabilities of a plasma filament. The stability conditions are presented and the frequency range of the rapidly varying stabilizing magnetic fields is determined.

IT has been shown earlier,[1] that a current-carrying plasma ring can have stable equilibrium in a magnetic field of definite configuration. The equilibrium condition given in reference 1, reduces to a relation between the intensity of the magnetic field on the equilibrium orbit and the average value of intensity of the field linked with the ring. The stability conditions impose limitations on the rate of decrease of the magnetic field along the radius in the region of the equilibrium orbit. These conditions are valid when the ring moves as a unit, and when its radius changes but its form remains circular. It is known, however, that a plasma current ring is unstable when its shape is distorted. In the present paper we consider the possibility of obtaining a ring which is stable under disturbances that distort the form of the ring, when the wavelength of the perturbations is sufficiently large compared with the radius of the ring cross section.

A ring with current is acted upon by an electrodynamic force made up of the forces due to the interaction between the current and the external magnetic field and the forces due to the interaction between the current and its own field. The first of these forces is of the form

$$F_1 = 2\pi R\,[\mathbf{J}\times\mathbf{H}]/c. \qquad (1)$$

The second force can be represented as the derivative of the potential function of the current $U = LJ^2/2c^2$ with respect to the corresponding coordinate. Then the component of this force, acting in the plane of the ring, will be

$$F_R = -\frac{\partial U}{\partial R} = \frac{2\pi J^2}{c^2}\left(\ln\frac{8R}{r_0} - 1\right). \qquad (2)$$

Here r_0 is the radius of the ring section, R is the radius of the ring, and H is the intensity of the field on the orbit.

If the ring is in equilibrium, there are no forces acting in the vertical plane along the z axis. Therefore the equilibrium condition reduces to the relation

$$F_1 + F_R = \frac{2\pi RJ}{c}\left[H + \frac{J}{cR}\left(\ln\frac{8R}{r_0} - 1\right)\right] = 0. \qquad (3)$$

Expressing J in terms of the flux linked with the ring, we arrive at Eq. (9) of reference 1

$$H = \frac{\overline{H}}{4}\left(1 + \frac{2}{l}\right), \qquad l = 2\left(\ln\frac{8R}{r_0} - 2\right). \qquad (4)$$

Here l is the inductance of one centimeter of length of the ring, and $\overline{\mathrm{H}}$ is the average value of the field inside the ring.

If the ring deviates little from the equilibrium position, the electrodynamic force acting in the plane of the ring can be represented in the form

$$(\partial F_1/\partial R - \partial^2 U/\partial R^2)\,\Delta R,$$

while the force acting in a plane perpendicular to the ring is given by

$$(\partial F_1/\partial z - \partial^2 U/\partial z^2)\,\Delta z,$$

where the values of F_1 and of $\partial U/\partial R$ correspond to the equilibrium position.

In order to introduce the force that arises when the ring is distorted, we use for simplicity the model of a straight current-carrying conductor. If the wavelength of the distortion is considerably less than the radius of the ring, such a model does not introduce any significant error. The expression for the force acting when a straight current-carrying conductor is bent, has the following form (see, for example, reference 2).

$$\frac{J^2}{c^2}\left(\frac{2\pi}{\lambda}\right)^2 \zeta \ln\frac{\pi\lambda}{r_0}, \qquad (5)$$

where ζ is the magnitude of the deviation, which is small compared with the wavelength of the distortion λ.

In addition to this force, the following forces will act in the plane of the ring

$$\frac{\partial F_R}{\partial R}\Delta R = \frac{2\pi J^2}{c^2 R}\Delta R, \qquad \frac{\partial F_1}{\partial R}\Delta R = \frac{2\pi J}{c}\left(H + R\frac{\partial H}{\partial R}\right)\Delta R, \qquad (6)$$

while the force

$$\frac{\partial F_1}{\partial z}\Delta z = -\frac{2\pi RJ}{c}\frac{\partial H}{\partial R}. \tag{7}$$

will act in the plane perpendicular to the ring. We use here the condition curl **H** = 0, and consequently $\partial H_R/\partial z = \partial H_z/\partial R$. Since all the expressions contain only the z component of the intensity of the magnetic field, we drop the subscript. It is clear that the fields and the derivatives of the fields with respect to coordinates are taken at the point corresponding to the equilibrium position.

We can now write the complete equations of motion of a ring that deviates from the equilibrium position. For the radial deviations we have

$$MN\frac{d^2\Delta R}{dt^2} = \left[\frac{J(H + R\partial H/\partial R)}{cR} + \frac{J^2}{c^2R^2} + \frac{J^2}{c^2}\left(\frac{2\pi}{\lambda}\right)^2 \ln\frac{\lambda}{\pi r_0}\right]\Delta R \tag{8}$$

and for vertical deviations we have

$$MN\frac{d^2\Delta z}{dt^2} = -\left[\frac{J}{c}\frac{\partial H}{\partial R} - \frac{J^2}{c^2}\left(\frac{2\pi}{\lambda}\right)^2 \ln\frac{\lambda}{\pi r_0}\right]\Delta z. \tag{9}$$

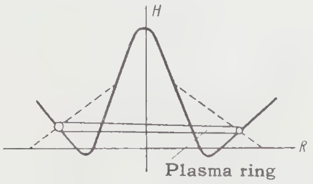
Here M is the mass of the ion and N is the number of the ions in the transverse section of the ring per unit length.

The value of the current J flowing through the ring in equilibrium is connected with the magnitude of the field by the relation

$$J = -2cRH/(l+2) \tag{10}$$

[See Eqs. (4) and (9) of reference 1].

If we substitute this expression into (8) and (9), it follows from the latter that when $2\pi R/\lambda > 1$, it becomes impossible to satisfy simultaneously the stability conditions in the horizontal and vertical planes. In order for the ring to be stable under deviations in the plane of the ring, the form of the field must be such as shown by the solid curve in the figure. The field near the equilibrium orbit should increase with increasing radius and its rate of increase should be the greater, the shorter the wavelength of the distortion which we wish to stabilize. On the other hand, to stabilize the ring in the plane perpendicular to the ring, the field must diminish with the radius, i.e., its shape should be approximately as shown dotted in the figure.



This contradiction can be resolved by using a method, which we shall call "dynamic stabilization." The shape of the field near the equilibrium state of the plasma ring is varied periodically in such a way, that at some definite instant of time it has the form shown in the figure by the solid line, and after half a cycle it changes to the form represented by the dotted line. In this case the expression for the field in the region of existence of the ring can be approximately written as

$$H = H_0 + [H_\sim + (R - R_0)(\partial H/\partial R)_\sim]\sin\omega t.$$

Here H_0 is the field component which is constant in space and in time, $H_\sim$ is the amplitude of the field which is constant in space but is alternating in time, $(\partial H/\partial R)_\sim$ is the amplitude of the rate of spatial change of the ac component of the field, R is the running coordinate, and R_0 is the radius of the equilibrium orbit.

It can be assumed that the magnitude of the current is determined essentially by the quasi-stationary, slowly varying component of the flux linked with the ring, while the rapidly varying component determines only the instantaneous value of the density of the field near the equilibrium orbit and affects little the value of the current. Then the equations of motion, for example Eq. (9), can be written in the form

$$\frac{d^2\Delta z}{dt^2} = \left[\frac{J^2}{MNc^2}\left(\frac{2\pi}{\lambda}\right)^2 \ln\frac{\lambda}{\pi r_0} - \frac{J}{MNc}\left(\frac{\partial H}{\partial R}\right)_\sim \sin\omega t\right]\Delta z. \tag{11}$$

This is none other than the equation of the small oscillations of an inverted pendulum on a vibrating suspension. The equation of motion of such a pendulum is

$$d^2\theta/dt^2 = (gL^{-1} - aL^{-1}\omega^2\sin\omega t)\,\theta, \tag{12}$$

where g is the acceleration due to gravity, L the length of the pendulum, a the amplitude of the oscillations of the point of suspension, and ω the angular frequency of the oscillations of the suspension.

The stability condition for systems described by such an equation is given by Jeffreys[3] and Kapitza:[4] Jeffreys solves the problem by the method of infinite determinants, while Kapitza uses the method of averaging. The results of their calculations are identical, and the stability condition reduces to the inequality

$$a^2\omega^2 \geqslant 2gL. \tag{13}$$

In order for the methods used in the investigation of (12) to be applicable, it is necessary to satisfy also the condition

$$a/L \ll 1, \tag{14}$$

which ensures the convergence of the solution.

On going over to the system described by

Eq. (11), conditions (13) and (14) reduce to

$$\frac{\omega^2 MN}{J} \geqslant \left(\frac{\partial H}{\partial R}\right)_{\sim} \geqslant \omega\,\frac{2\pi}{\lambda}\left(2MN \ln\frac{\lambda}{\pi r_0}\right)^{1/2}. \tag{15}$$

From this we obtain the condition for the frequency of the stabilizing field

$$\omega \geqslant \frac{4\pi v_T}{\lambda}\sqrt{\ln\frac{\lambda}{\pi r_0}}, \tag{16}$$

where the well-known relation

$$J^2 = 4c^2 N\varkappa T \tag{17}$$

is used, $v_T = \sqrt{2\kappa T/M}$ denotes the thermal velocity of the ions, and κ is Boltzmann's constant.

The condition for the rate of change of the magnetic field along the radius is written in the form

$$\left(\frac{\partial H}{\partial R}\right)_{\sim} \geqslant \frac{2J}{c}\left(\frac{2\pi}{\lambda}\right)^2 \ln\frac{\lambda}{\pi r_0}. \tag{18}$$

It follows from this expression that the dynamic stabilization is more effective for distortions with relatively long wavelengths, whereas the stabilizing action of a longitudinal stationary magnetic field is more effective for short-wave distortion (see, for example, reference 2). It seems natural therefore to use a relatively weak longitudinal stationary field to stabilize the short-wave distortions in conjunction with dynamic stabilization of the long-wave instabilities.

An analysis of the stability of the plasma ring under deformations in the plane of the ring [Eq. (8)] leads to relations similar to (16) and (17), except that in this case the only components of the magnetic field that can be effective are the one constant in space and the one constant in time. These components must be chosen in such a manner, that at the initial instant of time, upon formation of the discharge, the plasma ring is on an equilibrium orbit. Later on, in view of the finite active resistance of the plasma (see reference 5), the plasma ring will contract towards the center, but if the shape of the field is chosen, for example, as shown by the solid line in the figure, the plasma ring cannot go outside the region of positive values of H, and will be located, in the limit, near the zero value of H. Consequently the system described here permits, in principle, realization of an equilibrium loop over a time much longer than the time determined by the active resistance of the plasma.

Inequality (18) shows that in the case of dynamic stabilization the required intensity of the stabilizing field depends little on the degree of compression of the plasma, i.e., on the radius of the cross section of the ring, r_0, which enters only under the logarithm sign. It appears therefore that dynamic stabilization will make it possible to carry out investigations with a relatively larger particle density; consequently, relatively short lifetimes of the particle in the plasma are permissible here. This last circumstance may prove to be very important, since instabilities of other types, which appear in the quasi-stationary mode with low particle density (with the exception of the macroscopic instability considered here), may result in too large a disturbance to the magnetic thermal insulation of the plasma.

Instabilities of this kind are essentially connected with the deviation of the particle distribution function from equilibrium (Maxwellian) distribution and to the appearance of groups of particles with considerable ordered velocities. The action of rapidly varying fields on the plasma should prevent the development of effects which are due to the presence of ordered motions in the plasma.

The use of high-frequency fields in research on hot thermally insulated plasma entails considerable difficulties. These difficulties are due, on the one hand, to the need of constructing apparatus with power ratings much higher than those presently available. On the other hand, compared with quasi-stationary fields, the use of rapidly varying fields leads to additional energy losses in the external circuit. These energy losses are connected with the presence of skin effect, which makes the active resistance of the high frequency circuit, and consequently the losses in the circuit, assume an unusually large part in the general energy balance. Consequently the Q of the circuit in existing equipment, in the range of frequency of interest to us, cannot exceed several hundred. The generator power P necessary to supply a system with a given high-frequency circuit voltage, is given by the relation

$$PQ = \int\limits_V \frac{H^2}{8\pi}\,\omega dV, \tag{19}$$

where the integration is over the entire volume. If we neglect the dissipation and represent the working volume in the form of a torus with a large radius R and a small radius a, and use conditions (16) and (18), taking $(\partial H/\partial R)_{\sim}$ constant from zero to a, we arrive at the condition

$$P \geqslant \frac{\pi J^2}{c^2 Q}\, v_T\, \frac{R}{a}\left(\frac{2\pi}{\lambda}\right)^5 \left(\ln\frac{\lambda}{\pi r_0}\right)^{5/}\ .$$

This inequality, using (17), can be expressed in terms of the total number of particles in the working volume N and their temperature. Then

$$P \geqslant \frac{6NT^{3/2}}{aQM^{1/2}}\left(\frac{2\pi a}{\lambda}\right)^5\left(\ln\frac{\lambda}{\pi r_0}\right)^{5/2}\cdot 10^{-31}, \qquad (20)$$

where P is in megawatts and T in electron volts.

The following example illustrates the order of magnitude of the required power. Let us take gas — deuterium — with $M \approx 3 \times 10^{-24}$g, and let us stabilize the wavelengths greater than those for which $2\pi a/\lambda \approx 1$. A typical value is $\ln(\lambda/\pi r_0) \approx 2$; let there be $N = 10^{18}$ particles in the entire working volume and let a = 5 cm. Here

$$P \geqslant 0.5T^{3/2}/Q.$$

For an installation of this kind, which is feasible in the present technology, we assume as the characteristic values P = 50 megawatts and Q = 200. Then the temperature to which the plasma can be raised is 500 ev, which is fully adequate for experiments aimed at explaining the principal physical problems connected with this trend of research.

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ON PERTURBATION THEORY FOR LARGE QUANTUM SYSTEMS

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A new form of the Van Hove-Hugenholtz perturbation theory expansion is proposed. The series obtained differ from the known ones by the simplicity of the energy denominators; this can be useful, for instance, when evaluating excited states. The derivation of these expansions is also fairly simple.

IT is well known that the usual perturbation theory expansions cannot adequately describe systems with a large number of degrees of freedom which occupy a large volume in space. The main cause for this is that the series for such systems contain arbitrarily large powers of the volume.

At the present time this difficulty has already been overcome by various means. One of the different solutions of this problem is the theory proposed by Hugenholtz.[1] Using the formalism developed earlier by Van Hove[2,3] and amplifying his diagram method, Hugenholtz reorganized the series in such a way that the theory now only deals with quantities which are proportional to or independent of the volume. The new series contain, however, energy denominators that are more complicated than usual, which makes their practical application difficult. The energy of a perturbed system is, for instance, expressed in terms of the eigenvalues of the operator*

$$G(z) = (V + V(z - H_0 - G(z))^{-1} V + \ldots)_{id}. \qquad (1)$$

Here H_0 is the Hamiltonian of the unperturbed system, V the perturbation energy, and z a complex number.

Equation (1) is essentially an equation for $G(z)$ in the form of an infinite series. The series for other quantities are not in the form of equations, but they contain the same denominators $[z - H_0 - G(z)]^{-1}$.† This fact is particularly inconvenient in those cases where the problem to be solved is such that the series can not be broken off after a few terms, and one must sum (at least a well defined class of diagrams) in all perturbation theory orders. The case of a system with Coulomb interactions is, for instance, such a case.

We propose in the present note a simple derivation of the basic expansions of the Van Hove-Hugenholtz theory,[1-3] which leads to series with unperturbed denominators $(z - H_0)^{-1}$. The series obtained can also be used as the basis of the complete theoretical formalism of the papers mentioned, if this formalism is slightly changed (see the survey article[6]).

1. DERIVATION OF THE BASIC EXPANSIONS USING OPERATOR TECHNIQUES

We shall consider systems with a continuous (or quasi-continuous) spectrum, the Hamiltonian of which can be split into two parts

$$H = H_0 + V, \qquad (2)$$

where V can be considered to be a perturbation. We denote the wave function and eigenvalue of the unperturbed system by $|\alpha>_0$ and ϵ_α. We denote the same quantities for the perturbed system by $|\alpha>$ and E_α. We also introduce the perturbed and the unperturbed resolvents

$$R(z) = (z - H)^{-1}, \qquad D(z) = (z - H_0)^{-1}. \qquad (3)$$

In references 3 and 1 it was shown that*

$$|\alpha\rangle_\pm = R_{E_\alpha\pm}[R(z)|\alpha\rangle_0], \qquad E_\alpha = \varepsilon_\alpha + G_\alpha(E_\alpha). \qquad (4)$$

The symbol $R_{E_\alpha\pm}$ is defined by the equation

$$R_{E_\alpha\pm} f(z) = \lim_{z\to E_\alpha \pm i0} (z - E_\alpha) f(z), \qquad (5)$$

and the function $G_\alpha(E_\alpha)$ is the eigenvalue of the operator (1) in the point $z = E_\alpha$. It is clear that if $f(z)$ has a pole of the first order in the point $z = E_\alpha$, the limit (5) is uniquely defined and gives the residue of $f(z)$ in that pole. We retain the term "residue" also for the case where the two limits in (5) are different.

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*The index id indicates that only the so-called irreducible diagonal diagrams are taken into account when evaluating the matrix elements of the series (see reference 1).

†An exception is the series for the ground state, where the results can be simplified. This, however, is not so very interesting as the same results had been obtained earlier by Goldstone[4] and afterwards (and much more simply) by C. Bloch.[5]

*The presence of two signs for $|\alpha>$ corresponds to two complex-conjugate functions. This can easily be proved for a wide class of systems (see reference 6).

We must, according to Eqs. (4) and (5), expand $R(z) | \alpha >_0$ in V so that the factor that is singular at the point $z = E_\alpha$ is explicitly split off. We start from the easily verified formula

$$R(z) = D(z) + D(z) V R(z). \tag{6}$$

By iteration we get from (6) the series

$$R(z) = D(z) \sum_{k=0}^{\infty} (VD(z))^k, \tag{7}$$

which is useful for what follows, but which does not possess the property which we need at $E_\alpha \neq \epsilon_\alpha$. We return thus to (6). One can write any operator P as a sum of a diagonal and a non-diagonal operator (in the $|\alpha>_0$ representation)

$$P = P_d + P_{nd}. \tag{8}$$

We shall arrange the splitting up (8) in such a way that the second term on the right hand side of the expression for the matrix element,

$${}_0\langle\alpha | P | \beta\rangle_0 = P_{d\alpha}\delta(\alpha - \beta) + {}_0\langle\alpha | P_{nd} | \beta\rangle_0 \tag{9}$$

does not contain terms with a factor $\delta(\alpha - \beta)$. Here $P_{d\alpha}$ is the eigenvalue of the operator P_d and $\delta(\alpha - \beta)$ a product of the Kronecker symbol and the Dirac delta-function, if the quantum numbers of the system are partly discrete and partly continuous. In such a split, $(P_{nd})_d = 0$.

For the sake of simplicity we assume for the time being that

$$V_{nd} = V. \tag{10}$$

[The requirement (10) does not reduce the generality, as V_d can always be included in H_0. Practically speaking, however, it is inconvenient and we shall later on drop it.] Taking into account the fact that a product of operators of which only one is non-diagonal is itself a non-diagonal operator, we find the diagonal and the non-diagonal part of (6)

$$R_d = D + D(VR_{nd})_d, \tag{11}$$

$$R_{nd} = DVR_d + D(VR_{nd})_{nd}. \tag{12}$$

Assuming the series so obtained to converge, we solve (12) for R_{nd} by iteration. We get then

$$R_{nd} = K_{nd} R_d, \tag{13}$$

$$K_{nd} = \left\{ \sum_{k=1}^{\infty} (DV)^k \right\}_{\nu d}. \tag{14}$$

Here we have put, by definition,

$$(DV \ldots DVDV)_{\nu d} \equiv \{DV \ldots [DV(DV)_{nd}]_{nd}\}_{nd}. \tag{15}$$

By virtue of (11) we get from (13)

$$R_d = D + D(VK_{nd})_d R_d. \tag{16}$$

We introduce the diagonal operator*

$$G(z) \equiv (VK_{nd}(z))_d = \left\{ V \sum_{k=1}^{\infty} (D(z)V)^k \right\}_{sd}. \tag{17}$$

The meaning of the index sd is defined by Eq. (15) if the last index nd in that equation is replaced by d. Equation (16) now takes the form

$$R_d = D + DGR_d,$$

from which follows directly one of the basic Van Hove-Hugenholtz formulae[1,2]

$$R_d(z) = (z - H_0 - G(z))^{-1}. \tag{18}$$

For the resolvent we get

$$R(z) = \{1 + K_{nd}(z)\} R_d(z). \tag{19}$$

This is the second important formula of the same authors.

One can easily verify the validity of the relations

$$G^+(z) = G(z^*), \quad R_d^+(z) = R_d(z^*), \tag{20}$$

that follow from the hermiticity of H_0 and V.

Equations (18) and (19) have been obtained by relatively simple means and the denominators in the expansions for $K_{nd}(z)$ and $G(z)$ are in our case the operators $z - H_0$.

One verifies easily that the expansion (19), unlike (7), possesses the property we need and enables us to find in fact the residue (4). Indeed, by virtue of (14) the first factor on the right hand side of the equation

$$R(z) | \alpha\rangle_0 = \{1 + K_{nd}(z)\} | \alpha\rangle_0 R_{d\alpha}(z) \tag{21}$$

can suffer only a finite discontinuity on passing through the real axis. The factor $R_{d\alpha}(z)$, however, has, apart from a discontinuity on some segment of the real axis, also a pole in the point $z = E_\alpha$ [see (18)].

Up to now the derivation has been purely formal in character. For a practical application of the formulae obtained it is necessary to establish the rules for evaluating the matrix elements of operators with indices νd and sd. To understand these rules it is enough to consider an example. Let us assume that we must evaluate the matrix element

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*Van Hove had also obtained $G(z)$ in the form (17) by other means. The index sd (simple diagonal) was introduced by Van Hove (private communication).

$$ {}_0\langle\alpha|(VDV)_{\nu d}|\beta\rangle_0 = \int_{(\nu d)} d\alpha_1\, {}_0\langle\alpha|V|\alpha_1\rangle_0 D_{\alpha_1}\, {}_0\langle\alpha_1|V|\beta\rangle_0. \qquad (22) $$

According to (9) and (15), the index νd indicates that if $|\alpha\rangle_0 = |\beta\rangle_0$, the element (22) is evaluated by taking the limit $\alpha \to \beta$ for $\alpha \neq \beta$, while in the integral over α_1 we must exclude a small neighborhood near $|\alpha_1\rangle_0 = |\beta\rangle_0$ and let this excluded neighborhood tend to zero after the integral has been evaluated.

From the very beginning we could have started not from (6) but from the equivalent equation

$$ R = D + RVD; \qquad (6a) $$

and we should then, by analogous considerations, have been led to the equation

$$ R = R_d\{1 + Q_{nd}\}; \qquad (19a) $$

$$ Q_{nd}(z) = \left\{\sum_{k=1}^{\infty}(VD(z))^k\right\}_{\overline{\nu d}}, \qquad (14a) $$

$$ (VDVD\ldots VD)_{\overline{\nu d}} \equiv \{[(VD)_{nd}VD]_{nd}\ldots VD\}_{nd}. \qquad (15a) $$

Using the property $P^+_{nd} = (P^+)_{nd}$ one can verify that

$$ K^+_{nd}(z) = Q_{nd}(z^*). \qquad (23) $$

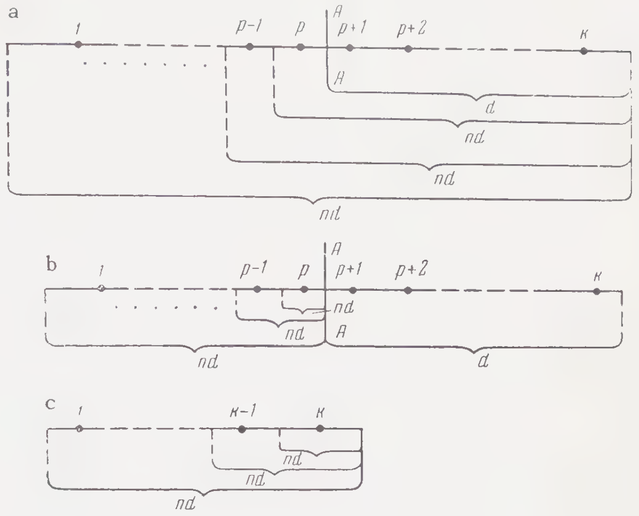
Equation (23) enables us to use Van Hove's method[2] to prove the important inequality

$$ \operatorname{Im} G_\alpha(z) \leqslant 0 \qquad (\operatorname{Im} z > 0). \qquad (24) $$

From the formulae established above we can obtain all basic results of Hugenholtz and Van Hove (see reference 6).

2. DERIVATION USING DIAGRAMS

In order to get rid of arbitrary powers of the volume in the expansions, we must be able to single out the singular factor not only from the resolvent as a whole, but also from its separate parts, which correspond only to diagrams of certain classes. [A method of comparing diagrams with the matrix elements of the operator (7) is described by Hugenholtz.[1]] For instance, it is important to separate the singularity of $R_L(z)$, which is that part of the resolvent obtained when only diagrams without vacuum components (linked clusters) are considered in the matrix elements of the series (7). This part of the resolvent is independent of the volume. The method considered in the foregoing for separating $R_d(z)$ from $R(z)$ cannot be applied immediately in this case. We give therefore a more general method, which is based upon the use of diagrams.



This derivation is essentially already suggested by the preceding arguments and leads also to expansions with simple denominators $z - H_0$. Condition (10) can now be dropped.

An arbitrary Hugenholtz diagram for $R_{nd}(z)$, $R_{Lnd}(z)$, or any other non-diagonal part of the resolvent has been schematically depicted in Fig. a*. Figure a is completely equivalent to Fig. b. In other words, if any diagram is cut by a vertical line AA successively after the first, second, . . . vertex, the diagram to the right of AA will be non-diagonal p-1 times, but the p-th time it will turn out to be diagonal. [Clearly (see Fig. c) this will be the largest diagonal part of the diagram, containing the right hand end.] Combining in one group all terms of the series with the same number p of vertices to the left of the diagonal part, and expanding the group in order of increasing p, we obtain easily not only Eq. (19), but also, for instance, the relation

$$ R_L(z) = \{1 + K_{Lnd}(z)\}\, R_{Ld}(z). \qquad (25) $$

Similarly we obtain also equations of the kind (17), (19a), and so on. In other words, equations which are completely analogous to the equations for $R(z)$ itself are also valid for parts of the resolvent.

We are now already able to reproduce all formulae from the Van Hove-Hugenholtz theory on the basis of the simplified expansions. In conclusion we note that the derivation with the aid of diagrams gives another procedure for evaluating the matrix elements of operators with νd and sd. To evaluate the elements of the operator $K_{nd}(z)$, for instance, we need only take into account diagrams of the type shown in Fig. c.

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*This figure is schematic in that the actual lines between vertices have been replaced by straight line segments, so as to guarantee the generality of our considerations.

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SOVIET PHYSICS JETP VOLUME 12, NUMBER 2 FEBRUARY, 1961

EFFECT OF WEAK INTERACTIONS ON THE ELECTROMAGNETIC PROPERTIES OF FERMIONS

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Self-mass effects and the singularities of behavior of fermions in a magnetic field, brought about by the universal Fermi interaction, are considered in the first order in the constant G_F. It is shown that parity nonconservation in weak interactions does not induce parity nonconservation in the field mass. An expression describing the effect of weak interactions on the electromagnetic properties of the fermions is found. The expression is proportional to the external current and is $\sim 10^{-2}$ times the similar expression obtained by taking into account the vacuum terms in electrodynamics.

IN the present note we consider the electromagnetic properties of fermions, brought about by the universal Fermi interaction, in the first order in e and G. The corresponding effects can, in principle, be verified experimentally in scattering processes and, by the same token, serve as a test for the existence of the universal interaction.

As is well known, writing of the Lagrangian of the Fermi interactions in the form of a simple product of two currents results in specific terms of the form

$$[p,\ n]^2 = \bar{p}\gamma_\mu(1+\gamma_5)n\cdot\bar{n}\gamma_\mu(1+\gamma_5)p,$$

$$[\nu,\ \mu]^2 = \bar{\nu}\gamma_\mu(1+\gamma_5)\mu\cdot\bar{\mu}\gamma_\mu(1+\gamma_5)\nu$$

and so forth. In particular, Zel'dovich[1] directed attention to the significance of these terms in scattering.

Let us first consider the term of local Fermi interaction of nucleons (the generalization to the case of other fermions presents no difficulties). Limiting ourselves to phenomenological account of the renormalization of the pseudovector current, due to the strong interaction, we can write the term $[p, n]^2$ in the Lagrangian in the form

$$L = \frac{G}{\sqrt{2}}\{[\bar{p}\gamma_\mu(1+\lambda\gamma_5)n,\ \bar{n}\gamma_\mu(1+\lambda\gamma_5)\ p]_+ + \zeta\ [\bar{p}\gamma_\mu\gamma_5 n, \bar{n}\gamma_\mu\gamma_5 p]_+\}, \tag{1}$$

in the absence of renormalization (the main case considered)

$$\lambda = 1, \qquad \zeta = 0. \tag{2}$$

Condition (2) can be satisfied for leptons; however, in this case ζ is never strictly equal to zero, because of electromagnetic effects. In general, (1) should also include terms with derivatives, but in the limit of a slowly varying electromagnetic field such an inclusion does not change the basic results obtained.

Making use of the relations put forth by Firz,[2] and taking commutation properties into account, we can rewrite Eq. (1) in the form*

$$\begin{aligned} L = -\frac{G}{\sqrt{2}}\Big\{&\frac{(1+\lambda)^2+\zeta}{4}\,[\bar{p}\gamma_\mu(1+\gamma_5)\,p\cdot\bar{n}^c\gamma_\mu(1+\gamma_5)\,n^c \\ &+\bar{n}\gamma_\mu(1+\gamma_5)\,n\cdot\bar{p}^c\gamma_\mu(1+\gamma_5)\,p^c] + \frac{(1-\lambda)^2+\zeta}{4} \\ &\times[\bar{p}\gamma_\mu(1-\gamma_5)\,p\cdot\bar{n}^c\gamma_\mu(1-\gamma_5)\,n^c \\ &+\bar{n}\gamma_\mu(1-\gamma_5)\,n\cdot\bar{p}^c\gamma_\mu(1-\gamma_5)\,p^c] \\ &+\frac{(1-\lambda^2)-\zeta}{2}\,[\bar{p}(1+\gamma_5)\,p\cdot\bar{n}^c(1-\gamma_5)\,n^c \\ &+\bar{p}^c(1-\gamma_5)\,p^c\cdot\bar{n}(1+\gamma_5)\,n]\Big\}. \end{aligned} \tag{3}$$

The terms of the Lagrangian (3) have the following general form:

$$K\psi_1 O^i\psi_1\cdot\bar{\psi}_2 O^i\psi_2, \qquad O^i = (1\pm\gamma_5), \quad (1\pm\gamma_5)\gamma_\mu, \tag{4}$$

where K is some coefficient. Here and below, summation is implied over the omitted vector indices of O^i. Now, repeating the reasoning carried out in the second half of our earlier paper,[3] we can show that the addition to the mass operator for particle 1, brought about by the presence in the Lagrangian of a component of the form (4), has, in first order in K, the form

$$\Delta M^1(x,\ x') = iKO^i\,\mathrm{Sp}\,[G^2(x,\ x')\,O^i]\,\delta(x-x'), \tag{5}$$

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*The following definition is employed: $\psi^C = \gamma_2 C\psi^*$, $C\gamma_\mu = \gamma^*_\mu C$, $C^+ = C^{-1}$; the asterisk denotes the complex conjugate.

where $G^2(x, x')$ is the causal propagation function of particles 2, which depends on the external field in the general case. Starting out from (5), one can write down the contribution to the operator of the field mass, for example, of the neutron, induced by the interaction (3). We have

$$\Delta M^n(x, x') = -\frac{G}{\sqrt{2}}\Big\{\frac{(1+\lambda)^2+\zeta}{2}\gamma_\mu(1+\gamma_5)$$
$$\times i\,\mathrm{Sp}\,[G^p(x, x')\gamma_\mu(1+\gamma_5)]$$
$$-\frac{(1-\lambda)^2+\zeta}{2}\gamma_\mu(1-\gamma_5)\,i\,\mathrm{Sp}\,[G^p(x, x')\gamma_\mu(1-\gamma_5)]$$
$$+\frac{(1-\lambda^2)-\zeta}{2}[(1+\gamma_5)\,i\,\mathrm{Sp}\,G^p(x, x')(1-\gamma_5)+(1-\gamma_5)$$
$$\times i\,\mathrm{Sp}\,G^p(x, x')(1+\gamma_5)]\Big\}\,\delta(x-x'), \tag{6}$$

where $G^p(x, x')$ is the proton propagation function. In the absence of an external field, the first two components of (6) yield a zero result because of the invariance conditions, while the remaining component leads to the diverging field mass

$$\Delta M^n = \frac{G}{\sqrt{2}\,(2\pi)^2}[(1-\lambda^2)-\zeta]\int\frac{(dp)\,M^p}{p^2+(M^p)^2} \tag{7}$$

or, if we introduce a cutoff in M,

$$\Delta M^n = GM^2[(1-\lambda^2)-\zeta]\,M^p.$$

The most important result is the fact that the field mass, in the case of a weak interaction, is the same for particles of different helicity i.e., parity nonconservation is not induced for free particles. This conclusion is not changed when the form of the Lagrangian (1) is made more accurate for strongly interacting particles, or when derivatives are introduced into (1).

We now consider the presence of an external electromagnetic field. In the expressions given for the field mass, it is necessary to substitute propagation functions that depend on the electromagnetic field. Here, taking account of the fact that actually $x = x'$ in (6), of the condition of gauge invariance, and of the equalities

$$\mathrm{Sp}\,\sigma_{\mu\nu} = \mathrm{Sp}\,\gamma_\alpha\sigma_{\mu\nu} = \mathrm{Sp}\,\gamma_5\gamma_\alpha\sigma_{\mu\nu} = 0,$$
$$\sigma_{\mu\nu} = (i/2)(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu),$$

we come to the conclusion that the static moment* induced by weak interactions is equal to zero (for a constant electromagnetic field $F_{\mu\nu}$).

In the case of a variable electromagnetic field, the terms induced by the weak interaction [1, 2][2] and dependent on the field do not vanish in the equations for the fermions 1 that interact with charged fermions 2. Starting out from (6), and making use of the expansion of the Green's function of the fermion in powers of the external electromagnetic field, which was given in the work of Karplus and Klein,[4] we obtain the result (after simple calculations) that a component

$$\Big\{\frac{eG_FM_2}{\sqrt{2}(4\pi)^2}[(1+\lambda+\zeta)+2\lambda\gamma_5]\int\limits_{M^{-2}}^{\infty}\frac{dS}{S}\,e^{-M_2^2S}\int\limits_{-1}^{1}\frac{(1-v^2)\,dv}{2}$$
$$\times\int(2\pi)^{-2}d^4k\;e^{ikx}\exp\Big[-S\,\frac{k^2(1-v^2)}{4}\Big]\,k_\alpha\gamma_\beta F_{\alpha\beta}(k)\Big\}\psi_1(x). \tag{8}$$

appears in the Dirac equation for particle 1. Here M is the cutoff parameter; as M → ∞, we obtain a logarithmic divergence. It is important that (8) always depends weakly on the choice of M.

For a more definite representation of the size of the effect, we put down the approximate value of the quantity in curly brackets in (8) keeping only the term with the lowest field derivative for the case of weakly interacting particles (2):

$$\Big\{\Big\}_1 = \frac{-5\sqrt{2}}{3(4\pi)^2}\,G_FM_2M_1\,\frac{e}{2M_1}(1+\gamma_5)\gamma_\alpha j_\alpha(x),$$
$$j_\alpha(x) = (2\pi)^{-2}\int d^4k\;e^{ikx}k_\beta F_{\alpha\beta}(k). \tag{9}$$

The existence of an effect corresponding to the appearance of the effective kinematic moment and described by the expression (8), can be judged by the neutrino (for $1 = \nu$, $2 = \mu$, e) or neutron (1 = n, 2 = p) behavior in a variable electromagnetic field. In the latter case we must separate the component of the effect that corresponds to violation of spatial parity [see the second term in the square brackets in (8)]. It is also interesting that, according to (8), ν_e and ν_μ ought to possess essentially different properties in the electromagnetic field.

In conclusion, we note that the presence of the universal nonlinear self-action of the fermions should lead to the appearance of an expression of the form (8), where G_F is replaced by a nonlinear constant and the particles 1 and 2 become identical.

The author expresses his thanks to D. D. Ivanenko for interest in the work. After completion of the present paper, the author became acquainted with the calculations of Ya. B. Zel'dovich and A. M. Perelomov, who obtained similar results by a different method. The author is grateful to Ya. B. Zel'dovich for discussions and for acquainting him with his results prior to publication.

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*Only the tensor Fermi interaction with particle 2, of the form $G'(\psi_1\gamma_\mu\gamma_\nu\psi_2\cdot\bar{\psi}_2\gamma_\mu\gamma_\nu\psi_1)$. could lead to an anomalous static magnetic moment μ_F^1 of particle 1. In this case, $\mu_F^1 = \frac{G'M_2M_1}{2\pi^2}\ln\frac{M}{M_2}\cdot\frac{e}{2M_1}$, where M is the cutoff parameter. Absence of a similar induced moment could serve as one of the indirect proofs of the exclusion of the tensor interaction.

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INSTABILITY OF A SYSTEM OF EXCITED OSCILLATORS WITH RESPECT TO ELECTROMAGNETIC PERTURBATIONS

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Possible mechanisms of autophase selection of excited oscillators in a radiation field leading to instability of the system with respect to electromagnetic disturbances are considered. It is shown that from the quantum point of view the instability of such systems may be due to unequal spacing in the (anharmonic) oscillator spectrum or to recoil during the emission of a photon.

1. It is known that a multi-velocity current of charged particles, moving in a straight line at a uniform velocity v greater than the velocity of light c_n in the surrounding medium is unstable under electromagnetic perturbations (see, for example, references[1-4]). From the classical point of view this instability can be attributed to the bunching (autophase selection) of the particles in the field of the electromagnetic wave that is propagated at an angle equal to the Cerenkov angle with respect to the direction of motion of the unperturbed current, and to the coherent Cerenkov radiation of the bunches formed. An analogous instability takes place also in a current of excited electric oscillators, with the one difference, that in this case, the autophase selection, which leads to the appearance of coherent radiation, turns out to be possible also for currents slower than light, $v < c_n$ (in particular, for $v = 0$).*

We consider here possible mechanisms of autophase selection of excited oscillators and offer an explanation of the instability of such a family of systems, both from the classical and from the quantum points of view.

2. We consider a current of excited oscillators, each of which comprises a charged particle, executing free oscillations of frequency ω_0 in the frame moving with the oscillators. We assume that initially the amplitudes of the oscillations of all the oscillators are identical, that the phases φ_k are distributed so that the alternating component of the average (macroscopic) current is everywhere zero, and that there is no electromagnetic radiation. Under the action of a randomly generated electromagnetic perturbation $\mathbf{e} = \mathbf{e}(\mathbf{r})e^{i\omega t}$, $\mathbf{h} = \mathbf{h}(\mathbf{r})e^{i\omega t}$, the motion of the oscillators is changed and an alternating polarized current appears, the radiation of which is superimposed on the initial perturbation.

Let us assume first that the oscillators are harmonic and that only a homogeneous (quasi-stationary) alternating electromagnetic field can exist in the system. It is not difficult to demonstrate that such a system will be stable. This is because the alternating component of the polarization current, by virtue of the linearity of the laws of motion of the oscillators, will be the same as in the system of unperturbed oscillators, which is stable when $v < c_n$. From this it follows that the phase sorting (bunching) of the excited oscillator under the action of the electromagnetic field, which is necessary for the instability, is possible only in those cases when the motion of the oscillators is nonlinear. By considering the equation of motion of an anharmonic oscillator in a sufficiently weak external field of frequency ω close to the frequency of free oscillation ω_0 (or close to one of its harmonics $p\omega_0$, $p = 1, 2, 3, \ldots$), we can distinguish two different mechanisms of phase sorting.

Phase bunching of anharmonic oscillators. A phase shift in the oscillations of an anharmonic oscillator may be caused by "phase instability" of the orbitally-stable motion of such an oscillator, which is a nonlinear conservative system.[8] The frequency of an anharmonic oscillator depends, generally speaking, on its energy E. Let, for example, $d\omega_0/dE < 0$. Then, those oscillators which, on average over a period, give up energy to the electromagnetic field (which is varying with a frequency $\omega \approx p\omega_0$) will oscillate more rapidly ($d\varphi_k/dt > 0$), than those which absorb energy ($d\varphi_k/dt < 0$); for $d\omega_0/dE > 0$ the sign of $d\varphi_k/dt$ is reversed. In either case, phase bunching of the excited oscillators occurs.*

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*In a current of electric oscillators (even unexcited), moving faster than light, there is a possibility of instability, which is related to the anomalous Doppler effect.[4,5] An example of such a system is an electron current moving in a straight line in a uniform magnetic field.[6] It is interesting to note that the possibility of generation and amplification of high-frequency oscillations in an electron beam system focused by a longitudinal magnetic field was indicated earlier by Pierce,[7] who, however, did not indicate the relation of this phenomenon to the anomalous Doppler effect.

We can explain the mechanism of phase sorting described above with an example of oscillators with one degree of freedom $x(t)$, in an alternating field $F_x = F_a(x, \dot{x})e^{i\omega t}$ of amplitude F_a. The general integral of unperturbed motion (for $F_x \equiv 0$) of such an oscillator is the periodic function

$$x_0(t) = x\{\omega_0(E_0)t + \varphi_0,\ E_0\}.$$

Perturbation of the constants of integration, $E = E_0 + \Delta E$ and $\varphi = \varphi_0 + \Delta\varphi$, leads to the appearance of an additional term $x_0^{(1)}(t)$ which equals, in the linear approximation ($|x_0^{(1)}| < |x_0|$)

$$x_0^{(1)} = \dot{x}_0(t)\Delta\varphi/\omega_0 + \{u(t) + M\dot{x}_0(t)t\}\Delta E, \qquad M = d\ln\omega_0(E)/dE, \tag{1}$$

where $u(t)$ is a periodic function of t. Equation (1) describes the "phase instability" of the orbitally-stable motion of an anharmonic oscillator: small perturbations of the energy (amplitude) lead to a phase shift that increases with time

$$\Delta\varphi_{eff} = \Delta\varphi + M\omega_0 t\Delta E.$$

Using (1) it is easy to find the perturbation $x^{(1)} = x - x_0$ ($|x^{(1)}| \ll |x_0|$) caused by the action of the high frequency field. In particular, for frequencies that coincide exactly, $\omega = p\omega_0$, we obtain

$$x^{(1)} = {}^1\!/_2\, MG_{-p}e^{-ip\varphi}\dot{x}_0(t)t^2 + \Psi_1(t)t + \Psi_2(t), \tag{2}$$

Where Ψ_1 and Ψ_2 are periodic functions of t, and G_k are the Fourier coefficients of the function

$$\dot{x}_0(\omega_0 t + \varphi,\ E_0)F_x(x_0,\ \dot{x}_0,\ t)\,\Delta(t) = \sum_k G_k \exp\{i(\omega + k\omega_0)t + ik\varphi\}.$$

Here $\Delta(t)$ is the Jacobian of the functions $\dot{x}_0(t)$ and $u(t) + M\dot{x}_0(t)t$ and is periodic in t. According to (2), the result of the prolonged action of a sufficiently weak field [when all the terms in (2) other than the first, may be ignored] is the appearance of a phase shift that increases with time

$$\Delta\varphi_{eff} \approx {}^1\!/_2\, MG_{-p}\omega_0 e^{-ip\varphi}t^2,$$

the sign of which depends on the initial phase of the oscillator φ.

Spatial bunching of oscillators moving in a non-uniform field. If the force acting on the oscillator depends on the coordinates and on the velocity, then, for $\omega = p\omega_0$, its mean value (over the period $2\pi/\omega$) will, generally speaking, not vanish and its sign will depend on the phase of the oscillator φ. As a result of the action of the generalized forces corresponding to the "external" degrees of freedom of the oscillators, a spatial regrouping of the oscillators occurs (displacement of centers of gravity or rotation of the direction of oscillation) in accordance with their phases.

For example, for a harmonic oscillator undergoing oscillations along the x axis [$x_0 = X_a\cos(\omega_0 t + \varphi)$] and capable of displacement along the z axis under the action of the force $f_z = F_a(x, \dot{x}, z, \dot{z})e^{i\omega t}$ of frequency $\omega = p\omega_0$ is given approximately by

$$z^{(1)} \approx {}^1\!/_2\, \dot{F}_{-p}e^{-ip\varphi}t^2, \tag{3}$$

where F_k are the Fourier coefficients of the force

$$F_a\{x_0(t),\ \dot{x}_0(t),\ z_0,\ 0\}e^{i\omega t} = \sum_k F_k \exp\{i(\omega + k\omega_0)t + ik\varphi\}.$$

The magnitude and sign of the displacement $z^{(1)}$ depend, according to (3), on the phase of the oscillator φ; it is this fact that ensures the spatial bunching of the oscillators according to their phase.

Both phase and spatial bunching of excited oscillators (in the general case they can exist simultaneously) lead to the appearance of a non-zero addition to the polarization current and consequently to additional coherent radiation (induced emission). For a rigorous investigation of the stability of a system of oscillators and a calculation of the increment that characterizes the rate of growth of the electromagnetic perturbation, it is clearly necessary to find the self-consistent electromagnetic field; the determination of this self-consistent solution requires specification of both the nature of the oscillators and the properties of the system that guides the radiation. The solution of some specific problems of such a type can be found in references 9 — 16. Here we limit ourselves to a simple energy calculation, which shows that an increasing electromagnetic field can occur in the system.

We consider, for example, a system of immobile anharmonic oscillators in a high-frequency field $\mathbf{e}(\mathbf{r})e^{i\omega t}$, $\mathbf{h}(\mathbf{r})e^{i\omega t}$ and calculate the work A, performed by the field on the oscillator. Using the perturbation method as before, and putting $\omega \gg |\omega - p\omega_0| \neq 0$, we obtain

$$x^{(1)} = -MG_{-p}\frac{\exp\{i(\omega - p\omega_0)t - ip\varphi)\}}{(\omega - p\omega_0)^2}\dot{x}_0(t) + O\{(\omega - p\omega_0)^{-1}\},$$

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*Such a phase sorting mechanism was considered by Agdur[9] for a quasi-stationary strophotron. A more general treatment is given in the work of Gaĭduk.[10]

and find for the mean power $P = \overline{dA/dt}^{\varphi} = \overline{\dot{x}F_x}^{\varphi}$, after averaging over the phases,

$$P = -M|G_{-p}|^2 \omega\alpha\gamma e^{2\alpha t}/(\alpha^2 + \gamma^2)^2 + O\{(\alpha^2 + \gamma^2)^{-1/2}\}, \tag{4}$$

where F_x is the x component of the Lorentz force acting on the oscillator, and $\alpha = -\text{Im}(\omega - p\omega_0)$, $\gamma = \text{Re}(\omega - p\omega_0)$.[*]

For mobile oscillators in a non-uniform field we obtain an analogous expression which differs from (4) only in a factor independent of α and γ. In this way, when $M \gtrless 0$ and $\gamma \gtrless 0$ the oscillators give up energy to the growing ($\alpha > 0$) oscillations. This indicates that such systems can be unstable with respect to electromagnetic excitation (induced emission prevails over absorption).

3. Although the instability of a system of excited oscillators is not related in essence to quantum effects and can be fully explained from the classical point of view, nontheless a quantum interpretation of the various mechanisms of instability may be useful. Moreover, a quantum "model" turns out to be, in a sense, more simple and graphic, if we are not concerned with a calculation of the increments that characterize the instability.[†] Bearing this in mind, we limit ourselves below to a qualitative consideration of simple idealized systems, neglecting questions concerning the calculation of the distribution of the oscillators over the levels, the finite line width of the radiation, etc.

(a) The instability of anharmonic-oscillator systems is related to the non-equidistant spacing in their energy spectra. Actually, let us assume that the oscillators, placed in an ideal resonator, are initially at identical levels ($N \neq 0$) and interact with one another only through the radiation field. Under these conditions a quantum emitted in the transition $N \rightarrow N - 1$ in a resonator tuned to a frequency $\omega = (E_N - E_{N-1})\hbar^{-1}$, cannot be absorbed in the transition $N \rightarrow N + 1$. As a result, owing to the large population of the level N, the energy of the field within the resonator will be increased, i.e., the system is unstable.[*] We note that a molecular generator, which employs oscillators with two levels (essentially nonlinear), is in this sense a special (but in principle, quantum) example of the systems considered.

(b) In the quantum approach, the recoil during the emission (or absorption) of a photon contributes to the displacement of a mobile excited oscillator in a non-uniform electromagnetic field. The picture of the instability in this case is more complicated, as each oscillator must have at least two degrees of freedom — "external," corresponding for example to translation along the z axis, and "internal" (vibration). We shall consider separately the interaction of such oscillators with traveling and with standing waves.

In investigating the interaction with traveling waves (wave vector **k**, refractive index of the medium n) it is natural to take as the initial state of the oscillators, which interact with each other only through the radiation field, a state with a definite z component of the total momentum $p_z^{(0)} = \mathbf{p}^{(0)}\mathbf{z}_0$ and a definite vibration energy E_N. From the laws of conservation of energy and momentum it follows, as is well known, (see, for example, reference 4) that in such a system there are, corresponding to the emitted and absorbed quanta, different frequencies, which are given by, for $\mathbf{k} \parallel \mathbf{z}_0$, $p_z^0 = 0$ and $\hbar\omega \ll E_N$

$$\hbar\omega_e = (1 - \hbar\omega n^2/2E_N)(E_N - E_{N-1}),$$
$$\hbar\omega_a = (1 + \hbar\omega n^2/2E_N)(E_{N+1} - E_N). \tag{5}$$

From (5) it is seen that even in the case of harmonic oscillators with equally spaced quantum levels, emission by the transition $N \rightarrow N - 1$ cannot be absorbed by the transition $N \rightarrow N + 1$.[†] This demonstrates the instability of the system with respect to moving electromagnetic waves.

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[*] In a reference frame where all oscillators move along the z axis with identical velocities v, it is necessary to allow for the Doppler effect. In the nonrelativistic case, if the question concerns interaction with a traveling wave $e^{i(\omega t - k_z z)}$, it is necessary to put in (4)

$$\alpha = -\text{Im}(\omega - k_z v - p\omega_0), \quad \gamma = \text{Re}(\omega - k_z v - p\omega_0).$$

[†] The situation is analogous for the instability associated with the anomalous Doppler effect.[3,4]

[*] For a finite oscillator radiation line width, finite Q of the resonator, and a distribution of oscillators over the initial states which is not a δ function, the absorption of the radiated quanta is, generally speaking, possible. Subsequent transitions of the oscillators to the levels $N \mp 2$, $N \mp 3$, etc., are also possible here. However, it is evident from qualitative considerations that in any case, for not too large line widths of the oscillators and the resonator, radiation will prevail over absorption after some time, i.e., instability is maintained [we note that when $E_{N+1} - E_N < E_N - E_{N-1}$ the instability may be at a frequency $\hbar\omega \lesssim E_N - E_{N-1}$, which agrees with the classical requirement that $\gamma < 0$ for $d\omega_0/dE < 0$, derived from (4). Highly characteristic, from this point of view, is the example considered below with relativistic electrons, where the non-equidistance of the energy levels is of the order $\hbar\omega_H/mc^2$, but instability is maintained as in the classical approximation.

[†] See the preceding footnote.

An interesting example of a system of mobile anharmonic oscillators is provided by relativistic electrons in a magnetic field H_0 - a rarefied nonequilibrium magnetoactive plasma with a δ distribution of parallel and transverse momenta p_z and $p_\perp$. It is easy to see that for such oscillators

$$(E_N - E_{N-1}) - (E_{N+1} - E_N) \approx (E_N - E_{N-1})^2/E_N \approx \hbar^2\omega_H^2/mc^2, \qquad (6)$$

where $\omega_H = eH_0c/E_N$ is the gyromagnetic frequency and $E_N \approx mc^2$. From (5) with (6) it follows that for $n = 1$ the Doppler effect, which determines the recoil when quanta are emitted in the z direction, compensates the unequal spacing of the levels of the relativistic electrons. In this case the emission and absorption occur exactly as in systems of immobile harmonic oscillators, and consequently the system (in the approximation considered) is stable with respect to waves, propagated along the magnetic field ($\mathbf{k} \parallel \mathbf{H}_0$). When $n \neq 1$ or $\mathbf{k} \neq \mathbf{H}_0$ there is no compensation and the system becomes unstable. Classical calculation confirms these considerations.[13,14,17]

We consider the interaction of mobile oscillators with standing waves in the simplest example, when the field in the system can be considered quasi-stationary, $\mathbf{E} = -\nabla\varphi$. The field of a two-dimensional quadrupole $\varphi = Aqxz$, where q is the coordinate of the field (charge on a capacitor),* is structurally the simplest nonuniform quasi-stationary field. The energy of interaction of such an oscillator (for example, an electron in a magnetic field $\mathbf{H}_0 = H_0\mathbf{z}_0$) with such a field can be written in the form $W \approx e\varphi = eAqxz$

We assume that the oscillator is vibrating in the plane xy (x oscillator) and for simplicity take the motion along the z axis also to be oscillatory (z oscillator), but with a low frequency ω_1. As is seen from the expression for the interaction energy, transitions of the x oscillators with absorption and emission of quanta, but without change of the state of the z oscillators, are forbidden in such a system. Therefore, if initially all the x oscillators are at the same level $N \neq 0$ but the z oscillators are at the lowest level, then emission and absorption proceeds at different frequencies.

$$\omega_e = (E_N - E_{N-1})\hbar^{-1} - \omega_1, \qquad \omega_a = (E_{N+1} - E_N)\hbar^{-1} + \omega_1.$$

In this way, even in the case of (linear) oscillators with an equidistant spectrum, excitation of the z oscillator (i.e., the displacement due to recoil during emission) makes the system unstable.

4. The considerations cited above make it possible to explain, both from the quantum and from the classical points of view, the instability of a whole series of systems containing excited oscillators. As already noted, the simplest example of an excited anharmonic oscillator is a charged particle moving with relativistic velocity along a helical trajectory in a uniform magnetic field $\mathbf{H}_0$, or along a trochoidal trajectory in crossed ($\mathbf{E}_0$ and $\mathbf{H}_0$) fields. Systems of such oscillators (electron currents in waveguides, unbounded inhomogeneous magnetoactive plasma) are analyzed in the classical approximation in references 11—14.* The analysis is analogous for other types of oscillators, for example electrons oscillating in an electrostatic potential well, rigid dipoles in a uniform field etc.[15-17] In all the particular cases, the solutions obtained in the classical approximation by the self-consistent field method show that such nonequilibrium systems are actually unstable with respect to electromagnetic excitations.

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RADIATION OF A PARTICLE MOVING ACROSS THE INTERFACE OF TWO MEDIA WITH ACCOUNT OF MULTIPLE SCATTERING*

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The spectral intensity distribution of the electromagnetic radiation emitted by a charged particle moving across the interface between two media is derived taking the effect of multiple scattering into account.

It was shown previously[1] that the transition radiation emitted by an ultra-relativistic particle in the direction of its motion can be greatly affected by multiple scattering in the region where the radiation originates. In the present article, formulas are given for the intensity of the electromagnetic radiation emitted by a particle entering or leaving a medium. The calculation is based on a simplified derivation of the transition-radiation equations, and on the method of taking into account the influence of multiple scattering on bremsstrahlung developed by Gol'dman.[2]

1. DERIVATION OF THE TRANSITION RADIATION FORMULAS

We shall demonstrate a simple method of deriving the formulas for the spectral intensity distribution of the transition radiation in certain interesting cases. As is well known, the energy radiated by a particle in the solid angle element $d\Omega$ in the frequency range $d\omega$ is given by the formula[3]

$$d\mathscr{E}_{\mathbf{n}\omega} = c\,|\mathbf{H}_\omega|^2 R^2 d\Omega d\omega, \qquad (1)$$

$$\mathbf{H}_\omega = e\frac{i\omega e^{i\omega R/c}}{2\pi c^2 R}\int_{-\infty}^{+\infty} e^{i(\omega t - \mathbf{kr}(t))}\,[\mathbf{n}\times\mathbf{v}(t)]\,dt. \qquad (2)$$

where R is the distance from the point of collision to the point of observation, $\mathbf{r}(t)$ and $\mathbf{v}(t)$ are the coordinate and the velocity of the particle respectively, and $\mathbf{k} = \mathbf{n}\omega/c$ is the wave vector of the emitted photon.

If the particle moves uniformly in a straight line in vacuum from $-\infty$ to $+\infty$, then the expression obtained by integrating Eq. (2) from $-\infty$ to 0 cancels the expression obtained by integrating from 0 to $+\infty$ and, as a result, no radiation is produced. If, in its motion, the particle goes from one medium into another (motion along the z axis, with the plane $z = 0$ separating the media), symmetry is lost and radiation appears.

It is evident that, at the interface of the media, radiation will be partially reflected and partially transmitted into the second medium. This radiation can easily be calculated and identified as due to the transition effect if it propagates in each of the media, after suitable refraction or reflection, from the point in which the trajectory intersects the interface between the media.

The transition radiation produced in such a process is directed both forward and backward at very small angles to the direction of motion of the particle. One can therefore expect such a calculation to yield a correct result for the case of an ultra-relativistic particle. Definite restrictions are then imposed on the angles and frequencies of the emitted radiation in each specific case. In particular, if vacuum is one of the media then, for the radiation into the vacuum, correct results are obtained for all frequencies since, in this case, $\vartheta \sim \sqrt{1-\beta^2}$.[4,5] For the "medium — medium" case, the formulas are applicable to radiation in the forward direction with frequencies greater than $\sqrt{\sigma}$, for which $\vartheta \sim \sqrt{\sigma}/\omega$ (reference 6) ($\sigma = 4\pi Ne^2/m$, $\epsilon = 1 - \sigma/\omega^2$).

It should also be noted that the wave vector $\mathbf{k}$ in Eq. (2) should be replaced in such problems, by $\mathbf{K}$, where $\mathbf{K}$ is a wave vector with a transverse component κ and a longitudinal component λ ($\lambda^2 = \omega^2\epsilon/c - \kappa^2$). As has been shown in reference 5, only such a choice of $\mathbf{K}$ permits a smooth transition of the fields at the interface of the media.

Let us now actually derive the transition radiation formulas. We shall first consider the case where the second medium is vacuum, and determine the radiation in that medium. We should use the vacuum value[6] of κ, equal to $\omega \sin \vartheta / c$, where ϑ is the angle between the direction of radiation and the z axis. Radiation emitted in the first medium will not all penetrate into the second. Accordingly, the magnetic field intensity in the first medium should be multiplied by the Fresnel coefficient for normal incidence $2\sqrt{\epsilon}/(1+\sqrt{\epsilon})$.[7] As a result, we have

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*Editor's note: Analogous qualitative results have been obtained independently by V. E. Pafomov, and are described in an article submitted to the editor on February 18, 1960.

$$H_\omega = e\frac{e^{i\omega R/c}\, v \sin\vartheta}{2\pi c^2 R}\left\{\frac{1}{1-\beta\sqrt{\varepsilon-\sin^2\vartheta}}\,\frac{2\sqrt{\varepsilon}}{1+\sqrt{\varepsilon}}-\frac{1}{1-\beta\cos\vartheta}\right\}. \tag{3}$$

Substituting Eq. (3) into Eq. (1), we obtain

$$d\mathscr{E}_{n\omega} = \frac{e^2\sin^2\vartheta}{4\pi^2 c}\left(\frac{\sqrt{\varepsilon}-1}{(1-\beta\cos\vartheta)(1-\beta\sqrt{\varepsilon-\sin^2\vartheta})}\right)^2 d\Omega\, d\omega, \tag{4}$$

i.e., a formula identical with that given in references 5 and 6 for the ultra-relativistic case.

If the first medium is vacuum, then the radiation emitted in it will be partially reflected from the interface of the medium. It is therefore necessary to multiply the field intensity by the Fresnel reflection coefficient for normal incidence $(1-\sqrt{\epsilon})/(1+\sqrt{\epsilon})$. The radiation emitted in the second medium may be neglected. Substituting $\kappa = \omega \sin\theta/c$, where θ is the angle between the direction of propagation of the radiation and the negative direction of the z axis, we obtain the following expression for the backward radiation:

$$d\mathscr{E}_{n\omega} = \frac{e^2\sin^2\theta}{4\pi^2 c}\,\frac{1}{(1-\beta\cos\theta)^2}\left(\frac{\sqrt{\varepsilon}-1}{\sqrt{\varepsilon}+1}\right)^2 d\Omega\, d\omega, \tag{5}$$

i.e., the Ginzburg-Frank[4] formula for the ultra-relativistic case.

Finally, let us consider the "medium – medium" case and find the radiation in the second medium with frequency $\omega \gg \sqrt{\sigma}$. We then have $\kappa = (\omega/c) \times \sqrt{\epsilon_2}\sin\vartheta$, and the Fresnel transmission coefficient can, with good accuracy, be taken as unity. As a result we obtain the formula

$$d\mathscr{E}_{n\omega} = \frac{e^2\sin^2\vartheta}{4\pi^2 c}\left(\frac{\sqrt{\varepsilon_1}-\sqrt{\varepsilon_2}}{(1-\beta\sqrt{\varepsilon_1-\varepsilon_2\sin^2\vartheta})(1-\beta\sqrt{\varepsilon_2}\cos\vartheta)}\right)^2 d\Omega\, d\omega, \tag{6}$$

which is identical with the usual formula for the forward transition radiation for $\beta \sim 1$ and $\omega \gg \sqrt{\sigma}$.

2. EFFECT OF MULTIPLE SCATTERING

It was shown by Pomeranchuk and the author[1] that, for particles moving with velocities close to that of light, multiple scattering influences only the forward transition radiation with frequencies $\omega \gg \sqrt{\sigma}$. It is therefore sufficient to assess the influence of multiple scattering on transition radiation within the framework of the approach described in Sec. 1. In addition, such an approach is convenient because the corresponding expressions are of a form suitable for the derivation of the bremsstrahlung formulas, taking multiple scattering into account, whether the particle moves in an unbound homogeneous medium[8,9] or whether it moves from a vacuum into a medium.[2] The latter case, in which the additional radiation originating near the medium boundary is considered, in addition to the bremsstrahlung with the effects of multiple scattering, has a direct bearing on our problem. This radiation is generated, because the particle moved in the vacuum to the left of the boundary without undergoing scattering.

Let the particle move from the vacuum into the medium in which it begins to undergo multiple scattering. According to the preceding section, in order to take transition radiation into account, it is necessary, in the derivation of Gol'dman,[2] to substitute for the wave vector **k** the quantity $\mathbf{K}_1$ $(\kappa_0\omega\sqrt{\epsilon}\sin\vartheta/c,\ \omega\sqrt{1-\epsilon\sin^2\vartheta}/c)$ in vacuum, and the quantity $\mathbf{K}_2(\mathbf{k}\sqrt{\epsilon})$ in the medium, where κ_0 is the unit vector in the plane perpendicular to the z axis.

In the frequency range $\omega \gg \sqrt{\sigma}$, it can be shown that the radiation associated with the transition of the particle through the medium boundary is given by the expression

$$E_\omega = \frac{e^2}{\pi c}\operatorname{Re}\left\{2\int_0^\infty \frac{dz}{1+z}\int_0^\infty (1+i)\,s' e^{-(1+i)(s'x+sz\,\mathrm{th}\,x)}\,dx + \int_0^\infty\left(\coth x-\frac{1}{x}\right)(1-(1+i)s'x)\,e^{-(1+i)s'x}\,dx - \ln\frac{s'}{s}-2\right\}, \qquad s=\tfrac{1}{4}(1-\beta^2)\sqrt{\omega/q},$$

$$s' = \tfrac{1}{4}(1-\beta^2+\sigma/\omega^2)\sqrt{\omega/q}, \qquad q=(c/4L)(E_s/E)^2, \tag{7}$$

where L is the radiation length and $E_s = 21$ Mev. Obviously, Eq. (7) includes both transition radiation and the radiation produced as the result of multiple scattering, i.e., bremsstrahlung. It is easy to obtain the following asymptotic expressions from Eq. (7):

$$E_\omega = \frac{e^2}{\pi c}\ln\frac{1}{s} \qquad (s' \ll 1); \tag{8}$$

$$E_\omega = \frac{e^2}{\pi c}\left[\frac{s'+s}{s'-s}\ln\frac{s'}{s}-2\right] \qquad (s' \gg 1). \tag{9}$$

As in other processes involving multiple scattering (see, e.g., the review by Feĭnberg[10]), one finds that there exists an energy $E_{cr} = (\mu c^2/E_s)^2 \times L\sqrt{\sigma}\mu c$ such that, for $E < E_{cr}$, multiple scattering does not affect the transition radiation and, consequently, there is also no additional bremsstrahlung radiation due to the existence of the boundary. To ascertain this, let us write the expression for s′ in the following form (for $\omega \ll \sqrt{\sigma}E/\mu c^2$):

$$s' = \frac{1}{2}\sqrt{\frac{E_{\mathrm{cr}}}{E}}\left(\frac{\sqrt{\sigma}\,E/\mu c^2}{\omega}\right)^{3/2}, \tag{10}$$

and for $\omega \gg \sqrt{\sigma}E/\mu c^2$ in the form

$$s' = \frac{1}{2}\sqrt{\frac{E_{\mathrm{cr}}}{E}}\left(\frac{\omega}{\sqrt{\sigma}\,E/\mu c^2}\right)^{1/2}. \tag{11}$$

If $E < E_{\mathrm{cr}}$, then it follows from Eqs. (10) and (11) that s' is always greater than unity, and the radiation is described by Eq. (9)

$$E_\omega = \frac{2e^2}{\pi c}\left[\left(\frac{1}{2} + \frac{\omega^2(1-\beta^2)}{\sigma}\right)\ln\left(1 + \frac{\sigma}{\omega^2(1-\beta^2)}\right) - 1\right], \tag{12}$$

i.e., the usual formula for the transition radiation. In the limiting cases of frequencies smaller and greater than $\sqrt{\sigma}E/\mu c^2$, we have, correspondingly,

$$E_\omega = \frac{e^2}{\pi c}\ln\frac{\sigma}{(1-\beta^2)\,\omega^2}, \tag{13}$$

$$E_\omega = \frac{e^2}{6\pi c}\left(\frac{\sigma}{(1-\beta^2)\,\omega^2}\right)^2. \tag{14}$$

For $E > E_{\mathrm{cr}}$ and $\omega < \sqrt{\sigma}E/\mu c^2$, it is convenient to introduce a characteristic frequency $\omega_1 = (\sigma E/E_s)^{2/3}(L/c)^{1/3}$. It can easily be seen that $s' \gg 1$ for $\omega \ll \omega_1$, and the radiation is given by Eq. (13); i.e., there is an agreement with the conclusion of reference 1 concerning the fact that, for $E > E_{\mathrm{cr}}$ and up to frequencies close to $\sim \omega_1$, the particle emits photons according to the usual formulas for transition radiation. For $\omega \gg \omega_1$, we have $s' \ll 1$, and, from Eq. (8), we find

$$E_\omega = \frac{e^2}{\pi c}\ln\left(\frac{2}{\sqrt{1-\beta^2}}\,\frac{E_s}{\mu c^2}\sqrt{\frac{c}{L\omega}}\right). \tag{15}$$

In considering the frequencies $\omega \gg \sqrt{\sigma}E/\mu c^2$, it is convenient to introduce another characteristic frequency $\omega' = (c/L)(E_sE/\mu^2c^4)^2$. For $\omega \ll \omega'$, $s' \ll 1$, and the radiation is described by Eq. (15). On the other hand, for $\omega \gg \omega'$ we have $s' \gg 1$, and Eq. (14) follows.

The energy E_{cr} in lead is equal to 400 Mev for electrons and 2.7×10^9 Bev for protons; in carbon, these energies are 7.6 Bev and 5×10^{10} Bev, respectively.

Thus, if for $E > E_{\mathrm{cr}}$ the usual transition radiation mechanism* begins to break down at $\omega > \omega_1$ because of multiple scattering, then the multiple scattering itself leads to the appearance of a "transition" bremsstrahlung and, as a result, we have the radiation described by Eq. (15), which already extends up to frequencies $\sim \omega' > \sqrt{\sigma}E/\mu c^2$. The region in which this radiation originates is of the order of $c/\sqrt{\omega q}$ (reference 2).

By means of analogous calculations, it can be shown that the radiation originating in the exit of a charged particle from a medium into a vacuum is, for frequencies $\omega \gg \sqrt{\sigma}$, described by formulas identical to those given above.

The author is indebted to I. I. Gol'dman for helpful comments.

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*As in reference 1, by "transition radiation" is meant the radiation due to a transition of a uniformly moving charge from one medium into another.[4,11]

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SOVIET PHYSICS JETP VOLUME 12, NUMBER 2 FEBRUARY, 1961

SECOND-ORDER EQUATIONS FOR SPINOR FIELDS

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Second order equations for fermions are investigated on the basis of the Lagrangian formalism. It is shown that an additional degree of freedom of the particles is due to two orientations of an imaginary electric dipole moment, which, however, may have a real radiation correction.

SECOND order equations for spinor wave functions have been proposed by Feynman and Gell-Mann[1] and have been discussed in different connections in a series of papers.[2-4] The present paper is devoted to the quantization of a spinor field satisfying a second order equation and to the interpretation of the additional degree of freedom.

Let the field be described by the spinor $\psi(x)$ with the usual transformation properties. Then we can write the Lagrangian which leads to the Klein-Gordon equation in the following form:

$$L = \frac{1}{m}\frac{\partial\bar{\psi}}{\partial x_\mu}\gamma_\mu\gamma_\nu\frac{\partial\psi}{\partial x_\nu} - m\bar{\psi}\psi, \tag{1}$$

$$m^{-1}(\Box - m^2)\psi(x) = 0. \tag{2}$$

On introducing the interaction with the electromagnetic field by means of the substitution $\partial_\mu \rightarrow \partial_\mu - ieA_\mu$, we can obtain the Lagrangian which leads to the equation of Feynman and Gell-Mann (provided that $\gamma_5\psi = \psi$).

For the time being we restrict ourselves to the discussion of the free field and calculate the dynamical variables in accordance with the usual rules:
the energy-momentum tensor

$$T_{\mu\nu} = \frac{1}{m}\frac{\partial\bar{\psi}}{\partial x_\rho}\gamma_\rho\gamma_\mu\frac{\partial\psi}{\partial x_\nu} + \frac{1}{m}\frac{\partial\bar{\psi}}{\partial x_\nu}\gamma_\mu\gamma_\rho\frac{\partial\psi}{\partial x_\rho} - \delta_{\mu\nu}L \tag{3a}$$

the current vector

$$J_\mu = \frac{i}{m}\bar{\psi}\gamma_\mu\gamma_\nu\frac{\partial\psi}{\partial x_\nu} - \frac{i}{m}\frac{\partial\bar{\psi}}{\partial x_\nu}\gamma_\nu\gamma_\mu\psi \tag{3b}$$

and the spin angular momentum tensor

$$S_{\lambda,\,\mu\nu} = \frac{i}{2m}\bar{\psi}\sigma_{\mu\nu}\gamma_\lambda\gamma_\rho\frac{\partial\psi}{\partial x_\rho} - \frac{i}{2m}\frac{\partial\bar{\psi}}{\partial x_\rho}\gamma_\rho\gamma_\lambda\sigma_{\mu\nu}\psi \tag{3c}$$

We note that $S_{0,0i} \neq 0$ $(i = 1, 2, 3)$, while in the theory based on the Dirac equation $S_{0,0i} \equiv 0$.

We now write $\psi(x)$ in the form of a superposition of plane waves:

$$\psi(x) = (2\pi)^{-3/2}\sum_{\lambda=1}^{4}\int\frac{d\mathbf{p}}{\sqrt{2E_p}}[a^\lambda(\mathbf{p})u(\mathbf{p})e^{-ipx} + b^{*\lambda}(\mathbf{p})v(\mathbf{p})e^{ipx}]. \tag{4}$$

Following the idea of Feynman and Gell-Mann we choose the basis spinors $u^\lambda(\mathbf{p})$ and $v^\lambda(\mathbf{p})$ to be the eigenfunctions of the projection operators $\frac{1}{2}(1 \pm \gamma_5)$ and $\frac{1}{2}(1 \pm \sigma_p)$ (here $\sigma_p = \boldsymbol{\sigma}\mathbf{p}/p$):

$$\begin{aligned}
&{}^1/_4(1+\gamma_5)(1+\sigma_p)u^1(\mathbf{p}) = u^1(\mathbf{p}),\\
&\quad {}^1/_4(1+\gamma_5)(1+\sigma_p)v^1(\mathbf{p}) = v^1(\mathbf{p}),\\
&{}^1/_4(1+\gamma_5)(1-\sigma_p)u^2(\mathbf{p}) = u^2(\mathbf{p}),\\
&\quad {}^1/_4(1+\gamma_5)(1-\sigma_p)v^2(\mathbf{p}) = v^2(\mathbf{p}),\\
&{}^1/_4(1-\gamma_5)(1+\sigma_p)u^3(\mathbf{p}) = u^3(\mathbf{p}),\\
&\quad {}^1/_4(1-\gamma_5)(1+\sigma_p)v^3(\mathbf{p}) = v^3(\mathbf{p}),\\
&{}^1/_4(1-\gamma_5)(1-\sigma_p)u^4(\mathbf{p}) = u^4(\mathbf{p}),\\
&\quad {}^1/_4(1-\gamma_5)(1-\sigma_p)v^4(\mathbf{p}) = v^4(\mathbf{p}).
\end{aligned} \tag{5}$$

If we assume that the operation of strong reflection of a spinor is carried out by means of the matrix $i\gamma_5$, then the following relation must hold

$$i\gamma_5 u^\lambda(\mathbf{p}) = \sigma_p v^\lambda(\mathbf{p}). \tag{6}$$

The spinors $u^\lambda(\mathbf{p})$ and $v^\lambda(\mathbf{p})$ can be expressed in terms of $u^\lambda(0)$ and $v^\lambda(0)$ by means of the Lorentz transformation matrix:

$$\begin{aligned}
u^\lambda(\mathbf{p}) &= \frac{E_p + p + m + (E_p + p - m)\gamma_5\sigma_p}{2\sqrt{m(E_p + p)}}u^\lambda(0),\\
v^\lambda(\mathbf{p}) &= \frac{E_p + p + m + (E_p + p - m)\gamma_5\sigma_p}{2\sqrt{m(E_p + p)}}v^\lambda(0).
\end{aligned} \tag{7}$$

Formulas (5) — (7) enable us to obtain the relations for normalization, orthogonality and summation over the index λ:

$$\bar{u}^\mu(\mathbf{p})u^{\lambda+2}(\mathbf{p}) = m\delta_{\mu\lambda}, \quad \bar{v}^\mu(\mathbf{p})v^{\lambda+2}(\mathbf{p}) = -m\delta_{\mu\lambda}; \tag{8a}$$

$$\sum_{\lambda=1}^{4}u^\lambda(\mathbf{p})\bar{u}^{\lambda+2}(\mathbf{p}) = mI, \quad \sum_{\lambda=1}^{4}v^\lambda(\mathbf{p})\bar{v}^{\lambda+2}(\mathbf{p}) = -mI; \tag{8b}$$

$$\sum_{\lambda=1}^{4} u^{\lambda}(\mathbf{p})\,\bar{u}^{\lambda}(\mathbf{p}) = \hat{p}, \qquad \sum_{\lambda=1}^{4} v^{\lambda}(\mathbf{p})\,\bar{v}^{\lambda}(\mathbf{p}) = \hat{p}. \tag{8c}$$

Here and later the index λ + 2 is equal to 1 and 2 for λ equal to 3 and 4.

We now consider the quantization of the free spinor field. From the canonical commutation relations it follows that

$$\begin{gathered}[\psi_{\alpha}(x),\ \bar{\psi}_{\beta}(y)]_{+} = -mi\delta_{\alpha\beta}D(x-y),\\ [\psi_{\alpha}(x),\ \psi_{\beta}(y)]_{+} = [\bar{\psi}_{\alpha}(x),\ \bar{\psi}_{\beta}(y)] = 0.\end{gathered} \tag{9}$$

These relations hold if

$$\begin{aligned}&a^{\lambda+2}(\mathbf{p})\,a^{*\mu}(\mathbf{q}) + a^{*\mu}(\mathbf{q})\,a^{\lambda+2}(\mathbf{p})\\ &\quad = b^{\lambda+2}(\mathbf{p})\,b^{*\mu}(\mathbf{q}) + b^{*\mu}(\mathbf{q})\,b^{\lambda+2}(\mathbf{p}) = \delta_{\lambda\mu}\delta(\mathbf{p}-\mathbf{q})\end{aligned} \tag{10}$$

and all the other anticommutators are equal to zero.

As may be seen from (10) the asterisks on the operators cannot denote Hermitian conjugation and, therefore, the norm of the state amplitude Ψ defined simply by $\Psi^*\Psi$ will not be positive definite. However, the operator η which has the properties

$$\begin{gathered}\eta = \eta^*,\ \eta^2 = 1,\ \eta a^{\lambda}(\mathbf{p})\,\eta = a^{\lambda+2}(\mathbf{p}),\\ \eta b^{\lambda}(\mathbf{p})\,\eta = b^{\lambda+2}(\mathbf{p}),\ \eta\Psi_0 = \Psi_0,\end{gathered} \tag{11}$$

generates a positive definite metric in Hilbert space of the state amplitudes: $\Psi_{\eta}^{*}\Psi > 0$; the observable quantities correspond to self-adjoint operators which now satisfy the condition $A = \eta^* A^* \eta$.

The operators for the energy-momentum, the charge, and the spin pseudovector are self-adjoint, as can be shown by utilizing relations (3), (4) and (8a). But the single particle states are also characterized by the imaginary eigenvalues ($\pm i/2$) of the component along the direction of motion of the polar vector $S_{0i} = \int S_{0,0i}dx$. This vector may be called the polar spin; with it is associated an imaginary dipole moment of the particle, which, however, may have a real radiation correction (if there exist interactions which are not invariant with respect to the combined inversion).

The Lagrangian for the interaction with the electromagnetic field has the form

$$L' = \frac{ie}{m}\left(\bar{\psi}\frac{\partial\psi}{\partial x_{\mu}} - \frac{\partial\bar{\psi}}{\partial x_{\mu}}\psi\right)A_{\mu} - \frac{e}{2m}\bar{\psi}\sigma_{\mu\nu}\psi F_{\mu\nu} + \frac{e^2}{m}A^2{}_{\rho}\bar{\psi}\psi. \tag{12}$$

By following the rules for constructing the scattering matrix in the case of derivative coupling,[5] one can prove the unitarity of the S matrix and carry out calculations for specific effects. In spite of its peculiar form which is close to the electrodynamics of scalar particles, the theory leads to the usual results. In Brown's paper[2] dealing with the Compton effect the approach to the problem is equivalent to ours if we assume that in nature there exist states corresponding only to "right-handed" (or "left-handed") particles. This assumption is not necessary; it may be shown (for example by considering Møller scattering) that the Pauli exclusion principle applies, as before, to states for which only the real quantum numbers coincide.

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SOVIET PHYSICS JETP VOLUME 12, NUMBER 2 FEBRUARY, 1961

LOCATION OF THE NEAREST SINGULARITIES OF THE $\pi\pi$-SCATTERING AMPLITUDE

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A number of $\pi\pi$-scattering diagrams are considered for which the so-called singular curves (threshold of appearance of imaginary part of the amplitude) have asymptotic values equal to 16 μ^2.

A knowledge of the singular points of amplitudes describing physical processes is of interest for its own sake as well as in connection with attempts to describe strong interactions phenomenologically.[1]

We shall find the singular points of the $\pi\pi$-scattering amplitude by making use of perturbation theory Feynman diagrams.[2] It is obvious that in order to find nearest lying singular points it is sufficient to consider only pions as the virtual particles.

The singular points corresponding to diagrams 1 and 2 (see Fig. 1)* were determined by Mandelstam.[3] Denoting by W the total energy of the incident pions in their barycentric frame, and by Q the momentum transfer we have:

$$Q^2 = 4W^2/(W^2 - 16) \text{ (diagram 1)} \tag{1}$$

$$Q^2 = 16W^2/(W^2 - 4) \text{ (diagram 2)} \tag{2}$$

(W and Q are expressed in units of pion mass).

In the $W^2 - Q^2$ plane the curves 1 and 2 have as asymptotes the straight lines $Q^2 = 4$, $W^2 = 16$ and $Q^2 = 16$, $W^2 = 4$. It is obvious from the pseudoscalar nature of the pion that the next boundary of singularities curves will have in the $W^2 - Q^2$ plane the asymptotes $Q^2 = 4$ and $W^2 = 36$ ($Q^2 = 36$, $W^2 = 4$) or $Q^2 = 16$ and $W^2 = 16$. We restrict ourselves to discussion of diagrams belonging to the latter class.

We were able to find only the four diagrams (3 – 6, Fig. 1), for which the singular curves have as asymptotes 16 and 16. Diagram 3 was studied by us previously.[4] It is convenient to express the curve corresponding to this diagram in parametric form

$$W^2(x, y) = 2(1 - xy)[1 - 2/(x - y)]^2,$$
$$Q^2(x, y) = 2(1 + xy)[1 + 2/(x + y)]^2, \tag{3}$$

with the two parameters x and y related by

$$x^2 = -1/y + 1 + y. \tag{3'}$$

Furthermore only the following values of x and y are permissible: $y > x$ and $y > -x$, as a consequence of the condition that the Feynman parameters α be positive. For $W^2 = Q^2$ we have $W^2 = Q^2 = 2(2 + \sqrt{5})^2 = 35.8$ and $x = 0$ while $y = (\sqrt{5} - 1)/2$.

The singular points for the diagrams 4 – 6 will be found by making use of the method described in detail previously.[4] The elementary, but occasionally tedious, steps will not be given. We make only two remarks. First, when the relation between Q^2 and W^2 is in parametric form it is convenient to use as parameters ratios of the Feynman parameters α, since the latter are positive and this facilitates the determination of the region of permissible values of the parameters. Second, by taking into account the symmetries in the Feynman diagrams, the determination of the location of the singularities is drastically simplified. This symmetry leads to the equality of the corresponding scalar products,* or, what is equivalent, to equalities among the α. We indicate below what equalities among the α resulting from the symmetry of Feynman diagrams were used by us.

For diagram 4 the symmetry leads to $\alpha_1 = \alpha_3$; $\alpha_2 = \alpha_4$; $\alpha_5 = \alpha_6 = \alpha_7 = \alpha_8$. The singular curve is given parametrically as follows:

$$W^2(x) = (3 + x)(9 - x^2)/(1 + x),$$
$$Q^2(x) = (3 - x)(9 - x^2)/(1 - x). \tag{4}$$

The parameter x is restricted to the range $-1 \leq x \leq 1$ by the condition that α be positive. Here $W^2(-1) = \infty$, $Q^2(-1) = 16$ and $W^2(1) = 16$, $Q^2(1) = \infty$. For $x = 0$ $W^2(0) = Q^2(0) = 27$.

For the diagram 5 the symmetry results in the equality $\alpha_2 = \alpha_4$. The singular curve is given by

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*In the diagrams referring to $\pi\pi$ scattering each pion is represented by a single line, while double and triple lines denote the exchange of two and three pions respectively.

*As shown by A. Z. Patashinskiĭ (private communication), one can verify directly that there are no nonsymmetric solutions for a majority of the diagrams considered here.

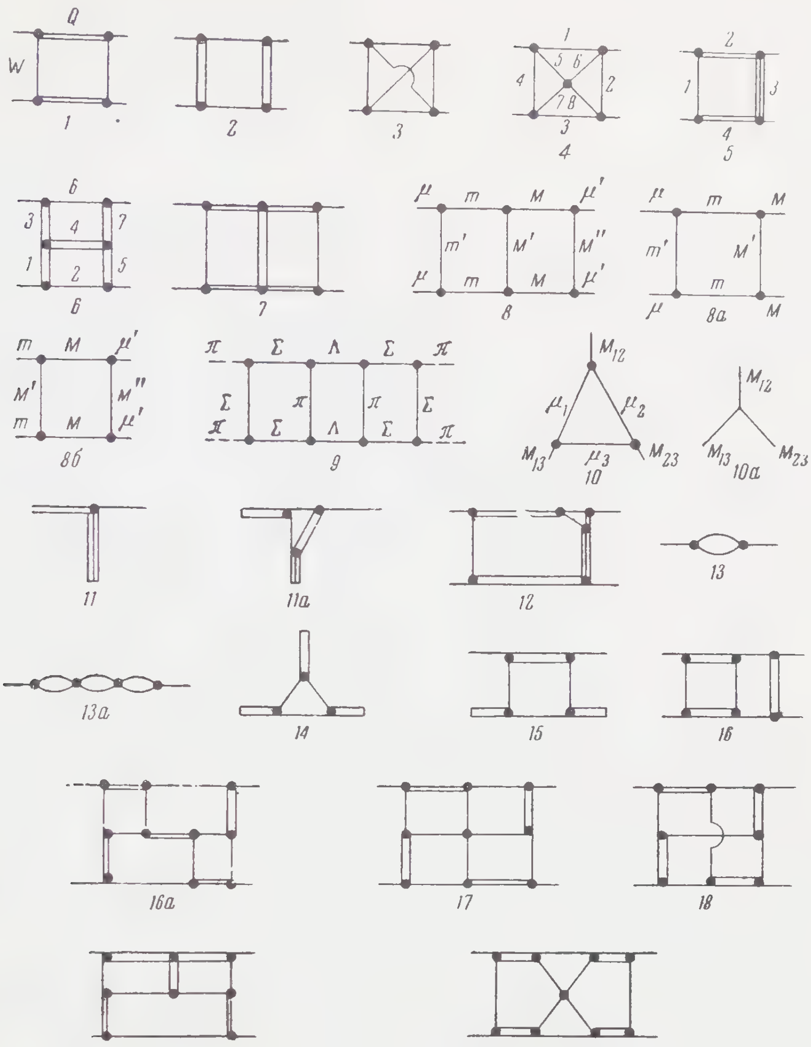

FIG. 1

$$(W^2 - 16)(Q^2 - 16) = 192. \tag{5}$$

It is obvious that the asymptotes are $W^2 = 16$ and $Q^2 = 16$. For $W^2 = Q^2$ we have $W^2 = Q^2 = 8(2 + \sqrt{3}) = 29.9$.

For the diagram 6 the symmetry leads to the equalities: $\alpha_2 = \alpha_6$; $\alpha_1 = \alpha_3 = \alpha_5 = \alpha_7$. The singular curve is given by

$$Q^2 = 16[\sqrt{W^2} - 1]^2 / \sqrt{W^2}\,[\sqrt{W^2} - 4]. \tag{6}$$

For $W^2 \rightarrow \infty$, $Q^2 \rightarrow 16$ and for $Q^2 \rightarrow \infty$, $W^2 \rightarrow 16$. For $W^2 = Q^2$ we have $W^2 = Q^2 = 34.6$.

The singular curves corresponding to the diagrams 1 — 6 are shown in Fig. 2 (curve 7 in Fig. 2 corresponds to diagram 7, which is obtained from 6 by the substitution $W^2 \rightleftarrows Q^2$).

As was already mentioned, we were unable to find any diagrams, other than diagrams 3 — 6, that would give rise to singular curves with asymptotes $Q^2 = 16$ and $W^2 = 16$. However we do not have a rigorous proof that no other such diagrams exist. We list below various considerations used by us in determining the absence of singular points in a number of diagrams.

First, we have assumed that it is sufficient to consider diagrams which when cut by an arbitrary horizontal or vertical line give a sum of the masses of the virtual particles equal to four. This assumption was based on the example of diagram 8 (the mass of the particle is given next to each line). In order that this diagram have a singular point it is necessary that diagrams 8a and 8b have singular points. If $\mu^2 < m^2 + m'^2$ and $M^2 < m^2 + M'^2$ then diagram 8a has a singularity only for $W^2 > 4m^2$. Analogously, if $m^2 < M^2 + M'^2$ and $\mu'^2 < M^2 + M''^2$ then diagram 8b has a singularity only for $W^2 > 4M^2$. Consequently diagram 8 has a singular curve with asymptotes $W^2 = 4\epsilon^2$ with ϵ the larger of the masses m and M.*

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*Proceeding in an analogous manner, it is easy to show that if the above conditions on the squares of the masses are violated either for diagram 8a or diagram 8b, then diagram 8 has no singularities at all; diagram 9 is an example of such a case.

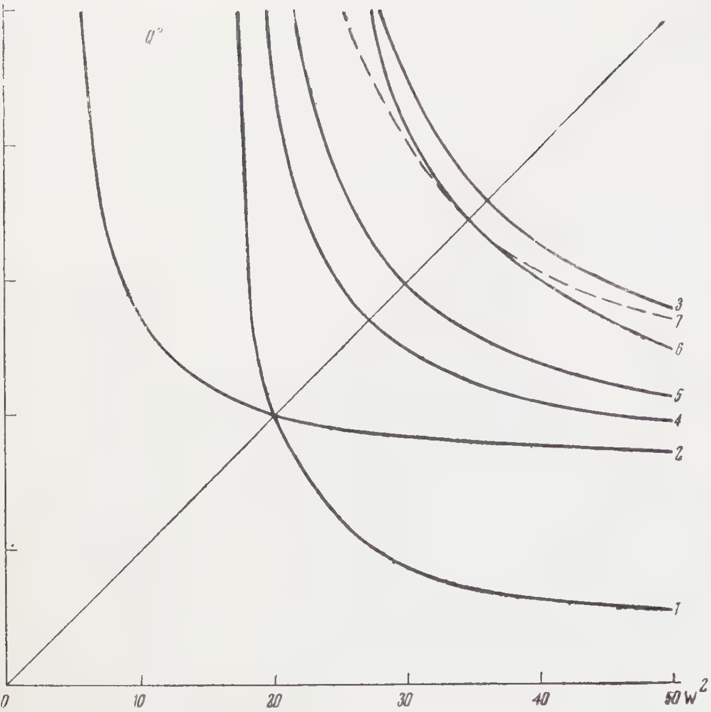

FIG. 2

Second, a substantial simplification in the analysis of complicated diagrams results when there is contained within them a three-sided contour such that the mass of the particle represented by the line entering one of the vertices is equal to the sum or the difference of the masses of the particles corresponding to the internal lines entering the same vertex. Diagram 10 is an example of such a contour. It is easy to verify that if $M_{12} = |\mu_1 - \mu_2|$, then it is necessary for the existence of a singular point to have $M_{13} = \mu_1 + \mu_3$ and $M_{23} = \mu_2 + \mu_3$. On the other hand, if $M_{12} = \mu_1 + \mu_2$, then $M_{13} = |\mu_1 \underset{\mp}{n} \mu_3|$ and $M_{23} = |\mu_2 \mp \mu_3|$. A three-sided contour satisfying these conditions can be transformed into a vertex (diagram 10a) and conversely the vertex of diagram 10a may be transformed into the three-sided contour of diagram 10.*

Let us discuss, for example, the vertex of diagram 11. It may be transformed into the three-sided contour, diagram 11a. In this manner it is possible to discuss along with diagram 5 diagram 12 which differs from diagram 5 by having one of the vertices satisfying the above conditions replaced by a three-sided contour. The location of the singularities of diagram 12 is the same as the location of the singularities of diagram 5. Applying to diagram 12 successive transformations of a vertex into a three-sided contour we obtain more and more new diagrams, all of which have singular lines coinciding with those of diagram 5. A situation arises that is analogous to what happens to the singularities of Green's functions: diagrams 13 and 13a have singularities in the same locations.

Let us also note the trivial circumstance that if one of the vertices of a three-sided contour satisfies the relation $M_{12} = |\mu_1 \pm \mu_2|$ while the other vertices fail to satisfy the corresponding relations, as for example is the case for diagram 14, then the presence of such a contour in a given diagram results in the absence of singularities.

Third, it is easy to establish that the four-sided contour of diagram 15 has no singular curves. As a result diagrams containing this four-sided contour also have no singularities (e.g., diagrams 16 and 16a).

Fourth, for a number of diagrams the absence of singularities was established by direct calculations (the impossibility of satisfying the condition that the parameters α be positive). Some of these diagrams (17—20) are shown in Fig. 1.

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*It is essential here that only one relation among the α exists in a three-sided contour.

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SPECTRA OF K_{e4} *AND* $K_{\mu 4}$ *DECAYS*

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We have obtained the electron and π-meson energy spectra and the $\pi - e$ angular correlation in K-meson decays $K \to 2\pi + e + \nu$, and also the μ-meson spectrum and the spectrum of the effective mass of the two π mesons in the $K \to 2\pi + \mu + \nu$ decay. The $K_{\mu 4}$ decay probability is one order of magnitude smaller than that of the K_{e4} decay.

IN a paper dealing with $K \to 2\pi + e + \nu$ decays, Okun' and the author[1] calculated the probability and obtained the effective-mass spectrum of the two π mesons. The present paper is devoted to a further study of K_{e4} decays, and also of $K_{\mu 4}$ decays. In Secs. 1 — 3 we calculate the energy spectra of the electrons and π mesons, and the angular $\pi - e$ distribution in the K_{e4} decay. In Secs. 4 and 5 we give the effective-mass spectra of two π mesons and the energy spectrum of the μ mesons in $K_{\mu 4}$ decay. The last section contains an expression for the K_{e4} and $K_{\mu 4}$ decay probabilities.

1. ENERGY SPECTRUM OF THE ELECTRONS IN K_{e4} DECAY

The relativistically-invariant amplitude of K_{e4} decay can be written in the form

$$M = \frac{\sqrt{2}G}{\sqrt{8E_K E_1 E_2}} \Big\{ f_1 Q_\sigma + f_3 q_\sigma + f_2 R_\sigma + \frac{f_4}{\mathfrak{M}^2} \varepsilon_{\sigma\rho\tau\eta} q_\rho (k_1)_\tau (k_2)_\eta \Big\} \bar{u}_e \gamma_\sigma (1 + \gamma_5) u_\nu, \tag{1}$$

where $k_1 = \{\mathbf{k}_1, E_1\}$ and $k_2 = \{\mathbf{k}_2, E_2\}$ are the 4-vectors of the π-meson momenta, q is the 4-momentum of the K meson, $Q = k_1 + k_2$, and $R = k_1 - k_2$; $\mathfrak{M}$ is a certain quantity with dimension of mass. The coefficients f_1 are functions of the scalar products $(k_1 q)$, $(k_2 q)$, and $(k_1 k_2)$. We shall assume them to be slowly-varying quantities and treat them as constants in the calculations. Then, if we use the rule $\Delta T = 1/2$ for lepton decays of strange particles, it is easy to verify that the first two terms in (1), which are symmetrical with respect to the substitution $k_1 \to k_2$, correspond to a decay in which the π mesons produced are in a state with total isotopic spin $T = 0$, while the remaining terms, which reverse their sign when the momenta k_1 and k_2 are interchanged, correspond to a state $T = 1$. Calculations show that the contribution of the last term of the total decay probability is related to the contribution of the second term, (which also corresponds to a π-meson system state with $T = 1$,) approximately as $(f_4^2/f_2^2) \times (m_\pi/\mathfrak{M})$.[4]

Inasmuch as the K-meson decay proceeds via $p - \bar{\Lambda}$ pair production,* we can regard the mass $\mathfrak{M}$ as equal to the baryon mass. Then the last term in (1) is small and will be neglected henceforth, and we shall consider only the vector part of the baryon current in the expression (1).

If we neglect the interaction of the π mesons in the final state, then f_1, f_2, and f_3 are real, and the differential probability has the form

$$dw = \frac{G^2}{2(2\pi)^8 E_K} [2(Vp_L)(Vp_\nu) - V^2(p_L p_\nu)] \times \delta^4(q - k_1 - k_2 - p_L - p_\nu) \frac{d^3k_1}{E_1} \frac{d^3k_2}{E_2} \frac{d^3p_L}{E_L} \frac{d^3p_\nu}{E_\nu}. \tag{2}$$

In the case of K_{e4} decay, the vector V contains only the π-meson momenta:

$$V = f(k_1 + k_2) + f_2(k_1 - k_2). \tag{3}$$

Integration of (2) is best carried out by introducing the variables $Q = k_1 + k_2$ and $R = k_1 - k_2$. Then we see from Eq. (A1) of the appendix that after integration over the π-meson momenta we obtain

$$\begin{aligned} dw = {} & G^2 M (2\pi)^{-5} \{ f^2 (1 + \cos\theta) Y^{1/2} \\ & + \tfrac{1}{3} f_2^2 M^{-2} [4M (M - E_e + E_e \cos\theta) \\ & - M^2 (1 + \cos\theta) - 4E_\nu (1 - \cos\theta) \\ & \times (M - E_e + E_e \cos\theta)] Y^{3/2} \} E_e^2 dE_e \, d\cos\theta \cdot E_\nu^2 dE_\nu, \\ Y = {} & [(M^2 - 2ME_e - 4m^2) / 2(M - E_e + E_e \cos\theta) - E_\nu] \\ & \times [(M^2 - 2ME_e)/2(M - E_e + E_e \cos\theta) - E_\nu]^{-1}, \end{aligned} \tag{4}$$

where M is the K-meson mass, m the π-meson mass, and θ the angle between the emission directions of the electron and of the neutrino. Integrating over E_ν between the limits

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*We use the Sakata model.[3]

$$0 \leqslant E_\nu \leqslant (M^2 - 2ME_e - 4m^2) / 2(M - E_e + E_e \cos\theta)$$

and then integrating with respect to $\cos\theta$ from -1 to 1, we obtain the electron spectrum, which is more conveniently expressed by introducing the variable $x = (M^2 - 2ME_e)/4m^2$, which depends linearly on E_e:

$$w(x)\,dx = \frac{G^2m^{10}}{3\pi^5M^3}\left(\frac{M^2}{4m^2} - x\right)^2$$
$$\times\left\{f^2\left[\left(x + \frac{5}{4} - \frac{3}{8x}\right)\sqrt{x(x-1)}\right.\right.$$
$$\left. - 3\left(x - \frac{1}{2} + \frac{1}{8x}\right)\ln(\sqrt{x} + \sqrt{x-1})\right]$$
$$+ \frac{f_2^2}{2}\left[\left(x + 6 - \frac{5}{8x} + \frac{3}{16x^2}\right)\sqrt{x(x-1)}\right.$$
$$\left.\left. - 3\left(2x + \frac{1}{4x} - \frac{1}{16x^2}\right)\ln(\sqrt{x} + \sqrt{x-1})\right]\right\}dx; \qquad (5)$$

the variation of x is within the limits $1 \leq x \leq (M/2m)^2$.

The distribution (5) has the form

$$w(x) = \frac{G^2m^{10}}{3\pi^5M^3}\,\frac{3}{8}\,[f^2F_0(x) + f_2^2F_1(x)], \qquad (5')$$

where the first term corresponds to a decay in which the π mesons are in a state with isotopic spin $T = 0$, while the second term corresponds to the case $T = 1$. The electron spectra (see Fig. 1) have in both cases a maximum at an electron energy ~ 70 Mev. The electron spectrum does not contain an interference term proportional to ff_2, because the term that describes the interference of the states of the system of π mesons of different parity drops out in the integration over the momenta of the π mesons.

The energy spectrum of the electrons in K_{e4} decay was also calculated by Chadan and Oneda.[4] The curves given in their article coincide with those obtained in the present paper.

2. ENERGY SPECTRUM OF π MESONS IN K_{e4} DECAY

In order to obtain the spectrum of the π mesons and at the same time the $\pi - e$ angular correlation, we integrate (2) with respect to d^3k_1 and d^3p_ν. We do this once more by an invariant method, introducing the 4-vectors

$$F = k_2 + p_\nu, \quad L = k_2 - p_\nu. \qquad (6)$$

Neglecting the electron mass, we can write V in the form

$$V = fq + f_2(q - 2k_1). \qquad (7)$$

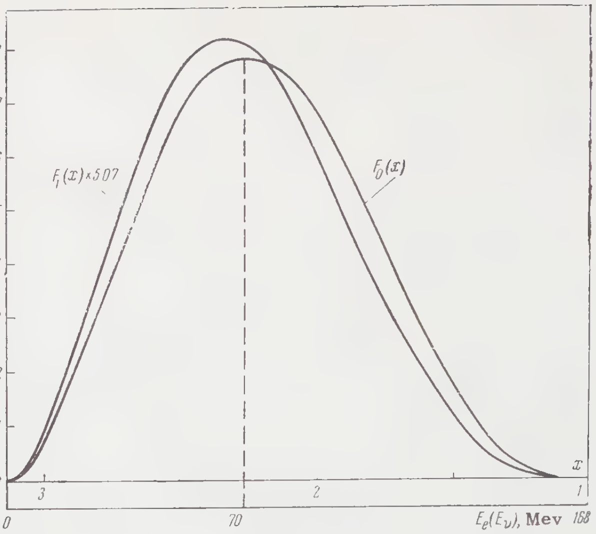

FIG. 1. Functions $F_0(x)$ and $F_1(x)$, which enter into formula (5) for the energy spectrum of the electrons in K_{e4} decay. The ordinate of the function F_1 is increased by a factor 5.07.

Substituting (7) in (2) and using Eq. (A2) of the appendix, we obtain

$$w(E_1, E_e, \cos\theta)\,dE_1\,d\cos\theta\,dE_e$$
$$= 4\pi^3G^2M(2\pi)^{-8}k_1\,dE_1\,d\cos\theta\cdot\{f^2[M - E_1 - k_1\cos\theta - 2E_e]$$
$$+ f_2^2[M - E_1 - k_1\cos\theta - 2E_e - 4M^{-2}(E_1 - k_1\cos\theta)$$
$$\times(M^2 - 2ME_1 + m^2 - 2E_e(M - E_1 + k_1\cos\theta))]$$
$$+ 2ff_2[M - E_1 + k_1\cos\theta + 2(m^2/M - E_1)$$
$$- 2E_e(2M^{-1}(M - E_1 + k_1\cos\theta) - 1)]\}$$
$$\times\left\{\left[\frac{M^2 - 2ME_1}{2(M - E_1 + k_1\cos\theta)}\right.\right.$$
$$\left.\left. - E_e\right] \Big/ \left[\frac{M^2 - 2ME_1 + m^2}{2(M - E_1 + k_1\cos\theta)} - E_e\right]\right\}^2 E_e^2\,dE_e. \qquad (8)$$

We integrate the resultant expression further with respect to the electron energy, between the limits

$$0 \leqslant E_e \leqslant (M^2 - 2ME_1)/2(M - E_1 + k_1\cos\theta)$$

and then with respect to $\cos\theta$ from -1 to 1. The π-meson spectrum expressed in terms of the variable $y = (M^2 - 2ME_1 + m^2)/m^2$ has the form

$$w_\pi(y)\,dy = \frac{G^2\pi^3m^7}{2(2\pi)^8}\,\frac{k}{m}\,\frac{1}{y^2}\left\{f^2\left[I_1y - I_2 + 2\frac{k^2}{m^2}\left(I_1 - \frac{2I_2}{3y}\right)\right]\right.$$
$$+ f_2^2\left[\left[1 - \frac{2m^2}{M^2}\left(\frac{M^2 - m^2}{m^2} - y\right)\right]\right.$$
$$\left.\times(I_1y - I_2) + 2\frac{k^2}{m^2}\left(I_1 - \frac{2}{3}\,\frac{I_2}{y}\right)\right]$$
$$+ 2ff_2\left[\left[y + \frac{m^2}{2M^2}\left(y^2 - \left(\frac{M^2 - m^2}{m^2}\right)^2\right)\right]I_1\right.$$
$$\left.\left. - I_2\frac{m^2}{M^2y}\left[y - \frac{1}{4}\left(y - \frac{M^2 - m^2}{m^2}\right)^2 - \frac{M^2k^2}{3m^4}\right]\right]\right\}dy, \qquad (9)$$

where $k = |\mathbf{k}|$ is the π-meson momentum;

$$I_1 = (y-1)^3/3 + 2y\,[2(y-1) - (y+1)\ln y],$$

$$I_2 = (y-1)^4/4 + (28y^3 - 27y^2 - 1)/6 - y^2(3-2y)\ln y; \tag{10}$$

y varies in the limits $1 \le y \le (M - m)^2/m^2$.

We represent (9) in the form

$$w(y) = \frac{G^2\pi^3 m^7}{2(2\pi)^8}\,[f^2\Phi_0(y) + f_2^2\Phi_1(y) + 2ff_2\Phi(y)]. \tag{9'}$$

The first term corresponds to K_{e4} decay in which the π mesons are in the state $T = 0$, the second corresponds to $T = 1$ and the third describes the interference of these states. The interference term is negative at large π-meson energies and positive at low energies, so that its integral vanishes. This is tantamount to stating that the total probability does not contain a term proportional to ff_2. The spectra of the π mesons (see Fig. 2) formed in the pure states $T = 0$ or $T = 1$ have broad maxima at approximately 25 Mev.

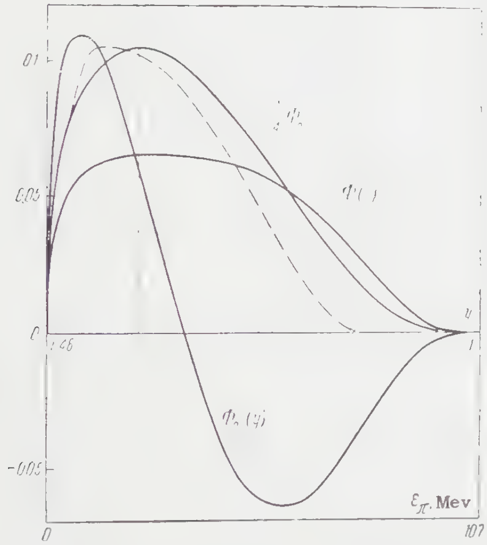

FIG. 2. The functions $(\frac{1}{4})\,\Phi_0(y)$, $\Phi_1(y)$, and $\Phi_{01}(y)$, which enter into the expression (9′) for the energy spectrum of the π mesons in K_{e4} decay; y varies in the limits $1 \le y \le 6.46$, corresponding to a π-meson kinetic-energy interval 107 Mev $\ge \varepsilon_\pi \ge 0$.

In the decay $K^+ \rightarrow \pi^+ + \pi^- + e^+ + \nu$, in which there is a mixture of the states $T = 0$ and $T = 1$, the presence of the interference term can cause a change in the form of the spectrum. In particular, if the sign of the interference term is negative, the maximum should shift towards larger π-meson energies. Therefore a study of the energy spectrum of the π mesons could possibly yield information on the relative signs of the constants f and f_2.

The spectrum of the π mesons in K_{e4} decay was also calculated by Mathur,[2] who used the nonrelativistic approximation. The π-meson spectrum obtained in the present work (and also the value of the maximum π-meson energy) differs from the result contained in Mathur's paper. For comparison, the spectrum obtained by Mathur for a decay in which the π mesons have $T = 0$ is shown dashed in Fig. 2.

3. ANGULAR π – E CORRELATION IN K_{e4} DECAY

The angular correlation between the directions of emission of the π meson and the electron is obtained from (8) by integration over E_e and then over E_1 — the π-meson energy. Integration over E_e yields

$$\begin{aligned}
&w(E_1, \cos\theta)\,dE_1\,d\cos\theta \\
&= \frac{G^2\pi^3 k_1 M m^6\,dE_1}{2(2\pi)^8}\Big\{f^2\Big[\frac{M - E_1 - k_1\cos\theta}{(M - E_1 + k_1\cos\theta)^3}\,I_1 \\
&- \frac{I_2 m^2}{(M - E_1 + k_1\cos\theta)^4}\Big] + f_2^2\Big[\frac{M - E_1 - k_1\cos\theta}{(M - E_1 + k_1\cos\theta)^3}\,I_1 \\
&- \frac{I_2 m^2}{(M - E_1 + k_1\cos\theta)^4} - \frac{4}{M^2}\,\frac{(E_1 - k_1\cos\theta)}{(M - E_1 + k_1\cos\theta)^3} \\
&\times((M - 2ME_1 + m^2)\,I_1 - I_2 m^2)\Big] \\
&2ff_2\Big[\frac{M - 3E_1 + k_1\cos\theta + 2m^2M^{-1}}{(M - E_1 + k_1\cos\theta)^3}\,I_1 \\
&- \frac{M - 2E_1 + 2k_1\cos\theta}{M(M - E_1 + k_1\cos\theta)^4}\,I_2 m^2\Big]\Big\}\,d\cos\theta.
\end{aligned} \tag{11}$$

Here I_1 and I_2 are determined by (10). Integrating (numerically) with respect to the energy E_1, we obtain the distribution over the angle between the directions of π and e. The angular distribution has the form

$$w(\cos\theta) = \frac{G^2\pi^3 m^7}{4(2\pi)^8}\,[f^2\varphi_0(\cos\theta) + ff_2\varphi_{01}(\cos\theta) + f_2^2\varphi_1(\cos\theta)]; \tag{11'}$$

and is shown in Fig. 3. It is seen from the figure

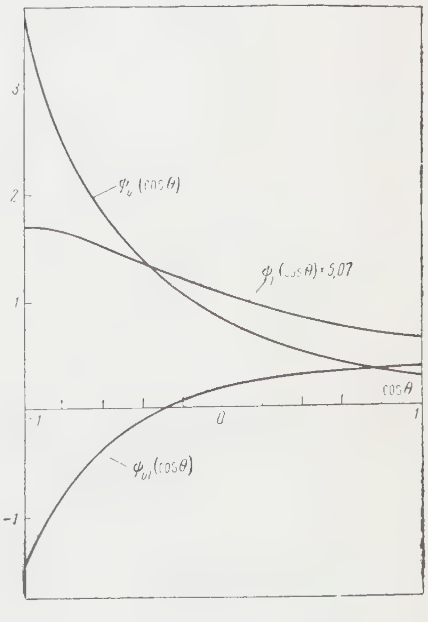

FIG. 3. Distribution over the angle between the directions of emission of the π meson and the electron (or neutrino) [Eq. (11′)].

that in the cases T = 0 and T = 1 the electron and the π meson are scattered predominantly at angles greater than $\pi/2$. Interference may change the form of the spectrum.

4. ANGULAR SPECTRUM OF μ MESONS IN $K_{\mu4}$ DECAY

In the case of $K_{\mu4}$ decay, the vector V in Eq. (2) contains three phenomenological constants:

$$V = f_1(k_1 + k_2) + f_2(k_1 - k_2) + f_3 q. \qquad (12)$$

As in the calculation of the electron spectrum in the K_{e4} decay, we introduce $Q = k_1 + k_2$ and $R = k_1 - k_2$ and integrate over $d^4Q\, d^4R$. We obtain

$$dw = \frac{G^2}{M(2\pi)^5}\Big\{f_1^2[M^2(E_\mu + p_\mu\cos\theta)$$
$$- 2M\mu^2 + \mu^2(E_\mu - p_\mu\cos\theta)]$$
$$+ f_3^2 M^2(E_\mu + p_\mu\cos\theta) + 2f_1f_3[M^2(E_\mu + p_\mu\cos\theta) - \mu^2 M]$$
$$+ \tfrac{1}{3} f_2^2[2M^2E_\mu - 2\mu^2 M$$
$$+ (E_\mu - p_\mu\cos\theta)(M^2 - 4ME_\mu + 3\mu^2)$$
$$- 4E_\nu(E_\mu - p_\mu\cos\theta)(M - E_\mu + p_\mu\cos\theta)]\frac{Q^2 - 4m^2}{Q^2}\Big\}$$
$$\times\left(\frac{Q^2 - 4m^2}{Q^2}\right)^{1/2} p_\mu\, dE_\mu\, d\cos\theta E_\nu^2\, dE_\nu, \qquad (13)$$

$$Q^2 = M^2 - 2ME_\mu + \mu^2 - 2(M - E_\mu + p_\mu\cos\theta)E_\nu. \qquad (14)$$

Integrating with respect to E_ν in the limits

$$0 \leqslant E_\nu \leqslant \frac{M^2 - 2ME_\mu + \mu^2}{2(M - E_\mu + p_\mu\cos\theta)}$$

and with respect to $\cos\theta$ in the limits $-1 \le \cos\theta \le 1$, we obtain the spectrum of the μ mesons in $K_{\mu4}$ decay. Expressed in terms of the variable $\varkappa = (M^2 + \mu^2 - 2ME_\mu)/4m^2$, this spectrum has the form

$$w(\varkappa)\,d\varkappa = \frac{G^2m^{10}}{3\pi^5M^3}\left\{\left[\left(\frac{M-\mu}{2m}\right)^2 - \varkappa\right]\left[\left(\frac{M+\mu}{2m}\right)^2 - \varkappa\right]\right\}^{1/2}$$
$$\times\Big\{f_1^2\varkappa\left(\frac{M^2-\mu^2}{4m^2} - \varkappa\right)J_1(\varkappa) + 2f_1f_3\varkappa\left(\frac{M^2+\mu^2}{4m^2} - \varkappa\right)J_1(\varkappa)$$
$$+ f_3^2\Big[\frac{M^2}{4m^2}\left(\frac{M^2-\mu^2}{4m^2} - \varkappa\right) - \left(\frac{(M-\mu)^2}{4m^2} - \varkappa\right)$$
$$\times\left(\frac{(M+\mu)^2}{4m^2} - \varkappa\right)\Big]J_1(\varkappa) + \frac{f_2^2}{2}\left(\frac{M^2-\mu^2}{4m^2} - \varkappa\right)J_2(\varkappa)\Big\}\,d\varkappa;$$

$$J_1(\varkappa) = \left(1 + \frac{5}{4x} - \frac{3}{8\varkappa^2}\right)\sqrt{\varkappa(\varkappa - 1)}$$
$$- 3\left(1 - \frac{1}{2\varkappa} + \frac{1}{8\varkappa^2}\right)\ln(\sqrt{\varkappa} + \sqrt{\varkappa - 1}),$$

$$J_2(\varkappa) = \left(\varkappa + 6 - \frac{5}{8\varkappa} + \frac{3}{16x^2}\right)\sqrt{\varkappa(\varkappa - 1)}$$
$$- 3\left(2\varkappa + \frac{1}{4\varkappa} - \frac{1}{16\varkappa^2}\right)\ln(\sqrt{\varkappa} + \sqrt{\varkappa - 1}). \qquad (15)$$

The variable $\varkappa$ ranges from 1 to $(M - \mu)^2/4m^2$. As $\mu \to 0$, the meson spectrum goes into the electron spectrum, with

$$(f_1 + f_3)^2 \to f^2, \quad f_2^2 \to f_2^2.$$

We can represent (15) in the form

$$w(\varkappa)\,d\varkappa = \frac{G^2m^{10}}{30\pi^5M^3}\{f_1^2F_0^{(1)}(\varkappa) + 2f_1f_3F_0(\varkappa)$$
$$+ f_3^2F_0^{(3)}(\varkappa) + f_2^2F_1^{(2)}(\varkappa)\}\,dx. \qquad (15')$$

Plots of the functions $F_\alpha^{(i)}$ are shown in Fig. 4.

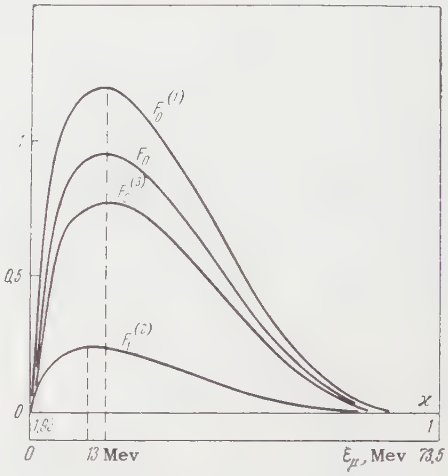

FIG. 4. The functions $F_\alpha^{(i)}(\varkappa)$, which enter into the expression (15′) for the energy spectrum of μ mesons in $K_{\mu4}$ decay ($\mathcal{E}_\mu$ is the kinetic energy of the μ meson).

The spectra have a maximum at a μ-meson energy 13 Mev.

5. EFFECTIVE MASS SPECTRUM OF TWO π MESONS IN $K_{\mu4}$ DECAY

The effective mass of the two π mesons is

$$Q_{eff} = \sqrt{Q^2} = \sqrt{(E_1 + E_2)^2 - (\mathbf{k}_1 + \mathbf{k}_2)^2}.$$

Substituting (12) in (2) and using the appendix formulas (A1) and (A3) we obtain

$$dw = \frac{G^2}{M(2\pi)^8}\sqrt{\frac{Q^2 - 4m^2}{Q^2}}\frac{(F^2 - \mu^2)^2}{F^4}\Big\{\frac{(f_1+f_3)^2}{3F^2}[(F^2 + 2\mu^2)$$
$$\times(QF)^2 - (2F^2 + \mu^2)Q^2F^2]$$
$$+ 2(f_1 + f_3)f_3\mu^2(QF) + f_3^2\mu^2F^2$$
$$+ \frac{f_2^2}{9}\frac{Q^2 - 4m^2}{Q^2}\Big[2\frac{F^2 + 2\mu^2}{F^2}(QF)^2$$
$$+ (4F^2 - \mu^2)Q^2\Big]\Big\}\sqrt{Q_0^2 - Q^2}\, Q\,dQ\,dQ_0\,d\Omega; \qquad (16)$$

here $F = q - Q$ and Q are 4-vectors. Integration over $d\Omega$ yields 4π, while integration over dQ_0 in the limits $Q \le Q_0 \le (Q^2 + M^2 - \mu^2)/2M$ leads to the following result: if we put $Q^2/M^2 = z$, then the spectrum over z is given by

$$w(z)\,dz = \frac{G^2M^7\pi^3}{6\,(2\pi)^8}\sqrt{1-4m^2/M^2z}\,\{(f_1+f_3)^2A(z)+f_3^2B(z)$$
$$+f_3(f_1+f_3)D(z)+\tfrac{1}{3}f_2^2[A(z)$$
$$+6zB(z)M^2/\mu^2]\,[1-4m^2/M^2z]\}\,dz. \qquad (17)$$

Here

$$A(z)=\tfrac{1}{8}[1-11\alpha-47\alpha^2-3\alpha^3+(-7+32\alpha+13\alpha^2)z$$
$$+(-7+\alpha)z^2+z^3]\sqrt{(1-\alpha+z)^2-4z}$$
$$+3\left[-\tfrac{3}{2}\alpha^2-\alpha^3+2\alpha^2z+(1-2\alpha+\alpha^2/2)z^2\right]\ln(\mathrm{I})$$
$$+3\alpha^2\left[-\tfrac{3}{2}-\alpha+(2z-z^2/2)\right]\ln(\mathrm{II}),$$

$$B(z)=\tfrac{1}{2}\alpha[1-5\alpha-2\alpha^2+5(2-\alpha)z$$
$$+z^2]\sqrt{(1-\alpha+z)^2-4z}-6\alpha[\alpha^2/2$$
$$+(1-2\alpha+\alpha^2/2)z+z^2]\ln(\mathrm{I})-3(1-z)\alpha^3\ln(\mathrm{II}),$$

$$D(z)=\alpha[1-5z-2z^2+5\alpha(2-z)$$
$$+\alpha^2]\sqrt{(1-\alpha+z)^2-4z}$$
$$+6\alpha[\alpha+\alpha^2-2\alpha z+(2-\alpha)z^2]\ln(\mathrm{I})$$
$$+6\alpha^2[\alpha+(1-z)^2]\ln(\mathrm{II}),$$

where $\alpha = \mu^2/M^2$ and we introduce the notation

$$\ln(\mathrm{I})=\ln\frac{2\sqrt{z}}{1-\alpha+z-[(1-\alpha+z)^2-4z]^{1/2}},$$

$$\ln(\mathrm{II})=\ln\frac{2\sqrt{z}\alpha}{(1-z)\,[(1-\alpha+z)^2-4z]^{1/2}-(1-z)^2-(1+z)\alpha}.$$

The effective-mass spectrum can be represented in the form

$$w(z)=\frac{10^{-3}\,G^2M^7\pi^3}{6\,(2\pi)^8}\left\{(f_1+f_3)^2A'(z)+3\frac{\mu^2}{M^2}f_3^2B'(z)\right.$$
$$\left.+3\frac{\mu^2}{M^2}f_3(f_1+f_3)D'(z)+\tfrac{1}{3}f_2^2H(z)\right\}. \qquad (17')$$

This spectrum is shown in Fig. 5. We see that the most probable decays are those in which the π mesons carry away a small effective mass, amounting to 0.6 M_K and nearly equal to twice the π-meson mass (0.565 M_K).

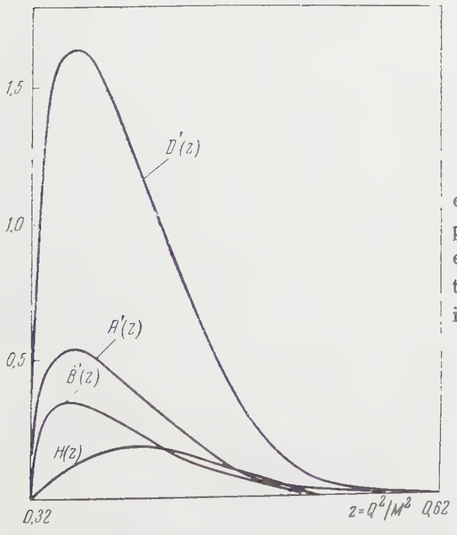

FIG. 5. Functions entering into the expression (17′) for the effective-mass spectrum of two π mesons in $K_{\mu 4}$ decay.

We note that the effective-mass spectrum of the π mesons produced in a state with isotopic spin T = 1 is shifted towards larger values of the effective mass. Accordingly, the electron spectra in K_{e4} decays and the μ-meson spectra in $K_{\mu 4}$ decays, pertaining to this case, have maxima at somewhat lower values of lepton energy than if the π mesons form a system with T = 0.

6. PROBABILITIES OF K_{e4} AND $K_{\mu 4}$ DECAYS

The matrix elements of the K_{L_2}, K_{L_3}, and K_{L_4} decays have the following form:

$$K^+_{e(\mu)\text{-}2}:\quad (G/\sqrt{2})\,\varphi_K\varkappa j^L_\alpha q_\alpha,$$
$$K^+_{e(\mu)\,3}:\quad (G/\sqrt{2})\,\varphi_K\varphi_\pi j^L_\alpha(g_1q_\alpha+g_2k_\alpha),$$
$$K^+_{e(\mu)\,4}:\quad (G/\sqrt{2})\,\varphi_K\varphi_\pi\varphi_\pi j^L_\alpha(f_1'k_1+f_2'k_2+f_3q)_\alpha, \qquad (18)$$

where $j^L_\alpha = \bar{u}_\nu\gamma_\alpha(1+\gamma_5)u_L$. For decays such as $K^+ \to N\pi^0 + e^+ + \nu$, where N = 0, 1, 2, these matrix elements become

$$N=0:\quad (G/\sqrt{2})\,\varphi_K j^L_\alpha\varkappa q_\alpha,$$
$$N=1:\quad (G/\sqrt{2})\,\varphi_K\varphi_\pi j^L_\alpha(g_1+g_2)\,q_\alpha,$$
$$N=2:\quad (G/\sqrt{2})\,\varphi_K\varphi_\pi\varphi_\pi j^L_\alpha[(f_1'+f_2')/2+f_3]\,q_\alpha. \qquad (19)$$

From the known probabilities of $K_{\mu 2}$ and K_{e3} decays we can obtain

$$(g_1+g_2)^2/\varkappa^2\approx 10^{-1}/0.14\cdot 10^{-2}\,M_p^2\approx 72/M_p^2 \qquad (20)$$

(M_p is the proton mass).

Assuming that each "excess" π meson introduces the same dimensional factor (20), i.e., assuming (for the K_{e4} and K_{e3} decays)

$$[(f_1'+f_2')/2+f_3]^2/(g_1+g_2)^2=(g_1+g_2)^2/\varkappa^2=72/M_p^2,$$

we obtain for the branching ratio of the K_{e4} and K_{e3} decays a value

$$w(K^+\to 2\pi^0+e^++\nu)/w(K^+\to\pi^0+e^++\nu)\approx 0.8\cdot 10^{-3}.$$

The probability of $K_{\mu 4}$ decay is

$$w_{K_{\mu 4}}=\frac{G^2m^{10}}{3\pi^5M^3}[0.0334\,f_1^2+2\cdot 0.041\,f_1f_3$$
$$+0.051\,f_3^2+0.0046\,f_2^2].$$

If we let the μ-meson mass approach zero, we obtain for the probability of K_{e4} decay

$$w_{K_{e4}}=(G^2m^{10}/3\pi^5M^3)\,[0.3\,(f_1+f_3)^2+0.06\,f_2^2].$$

It is thus found that the probability of $K_{\mu 4}$ is one order of magnitude smaller than the probability of K_{e4} decay.

In conclusion, the author expresses his gratitude to L. B. Okun' for suggesting the problem, for discussion, and for interest in the work. The author is also grateful to the associates of the Mathematics Division, V. A. Potapov, R. A. Ioffe, and L. I. Panov for performing the numerical calculations.

APPENDIX

In the integration over k_1 and k_2 (the momenta of the π mesons) we introduce the variables $Q = k_1 + k_2$ and $R = k_1 - k_2$. Then

$$\frac{d^3k_1}{E_1}\frac{d^3k_2}{E_2} = \delta(QR)\,\delta(Q^2 + R^2 - 4m^2)\,d^4Q\,d^4R,$$

$$\int \delta(QR)\,\delta(Q^2 + R^2 - 4m^2)\,d^4R = 2\pi\,[(Q^2 - 4m^2)/Q^2]^{1/2},$$

$$\int (RB)^2 \delta(QR)\,\delta(Q^2 + R^2 - 4m^2)\,d^4R$$

$$= \frac{2\pi}{3}\,[(QB)^2 - Q^2B^2]\left(\frac{Q^2 - 4m^2}{Q^2}\right)^{3/2}, \tag{A1}$$

where B is any 4-vector.

Introducing the vectors $F = k_2 + p_\nu$ and $L = k_2 - p_\nu$, we obtain

$$\frac{d^3p_\nu}{E_\nu}\frac{d^3k_2}{E_2} = \delta(F^2 + L^2 - 2m^2)\,\delta(FL - m^2)\,d^4F\,d^4L,$$

$$\int \delta(F^2 + L^2 - 2m^2)\,\delta(FL - m^2)\,d^4L = 2\pi\,(F^2 - m^2)/F^2,$$

$$\int (Bp_\nu)\,\delta(F^2 + L^2 - 2m^2)\,\delta(FL - m^2)\,d^4L = \pi\,(BF)\left(\frac{F^2 - m^2}{F^2}\right)^2. \tag{A2}$$

We denote by F and L the 4-vectors $F = p_\mu + p_\nu$ and $L = p_\mu - p_\nu$, where p_μ and p_ν are the 4-momenta of the μ meson and the neutrino. Then

$$\frac{d^3p_\mu}{E_\mu}\frac{d^3p_\nu}{E_\nu} = \delta(FL - \mu^2)\,\delta(F^2 + L^2 - 2\mu^2)\,d^4L\,d^4F,$$

$$\int \delta(F^2 + L^2 - 2\mu^2)\,\delta(FL - \mu^2)\,d^4L = 2\pi\,(F^2 - \mu^2)/F^2,$$

$$\int (NL)(ML)\,\delta(F^2 + L^2 - 2\mu^2)\,\delta(FL - \mu^2)\,d^4L$$

$$= 2\pi\,\frac{F^2 - \mu^2}{F^2}\left\{\mu^4\,\frac{(FN)(FM)}{F^4} + \frac{(F^2 - \mu^2)^2}{3F^4}\,[(FN)^2 \right.$$

$$\left. - F^2N^2]^{1/2}\,[(FM)^2 - F^2\mu^2]^{1/2}\right\}. \tag{A3}$$

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THE SCATTERING OF SPIN WAVES AND PHONONS BY IMPURITIES IN FERROMAGNETIC DIELECTRICS

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The effect of impurities on the heat conductivity of a ferromagnetic dielectric at low temperatures is examined. The conductivity of a ferromagnetic dielectric containing impurities is calculated.

THERMAL energy can be carried in ferromagnetic dielectrics by spin waves and by phonons. Akhiezer and Shishkin[1] have calculated the thermal conductivity determined by the interaction of spin waves and phonons with one another, and also by the interaction of spin waves with phonons. According to these authors, the conductivity varies exponentially with temperature at low temperatures. This relation is connected with the fact that it is only possible to obtain a finite value for the conductivity of a crystal if umklapp processes are considered, as was shown by Peierls.[2] In these, the quasi-momentum and consequently also the energy of at least one of the interacting particles must be sufficiently great.

If a crystal contains impurities, the scattering of long-wave spin waves and phonons by the impurities must be taken into account in calculating the thermal conductivity. Since there are considerably more long-wave phonons and spin waves than short-wave ones at low temperatures, the low-temperature conductivity will be determined by scattering of spin waves and phonons by impurities, and not by umklapp processes, the probability of which falls off exponentially. We shall not consider the excitation of optical-mode phonons, since their contribution to the thermal conductivity at sufficiently low temperatures is also exponentially small.

The aim of the present work is to study the influence of impurities on the thermal conductivity of ferromagnetic dielectrics. We can write the Hamiltonian for a ferromagnetic dielectric in the form

$$H = H^{(s)} + H^{(l)}, \tag{1}$$

where the first term is the exchange energy and the second is the energy of elastic lattice vibrations.

The exchange energy of a ferromagnetic dielectric containing impurities is

$$H^{(s)} = -\tfrac{1}{2}\sum_{l,n} J_1(R_{ln})\,\mathbf{s}_l\mathbf{s}_n - \sum_{l,n} J_{12}(R_{ln})\,\mathbf{s}_l\boldsymbol{\sigma}_n - \tfrac{1}{2}\sum_{l,n} J_2(R_{ln})\,\boldsymbol{\sigma}_l\boldsymbol{\sigma}_n, \tag{2}$$

where $\mathbf{s}_l$ is the spin of an atom of the basic (ferromagnetic) substance, $\boldsymbol{\sigma}_l$ is the spin of a paramagnetic impurity atom, $J(R_{ln})$ are the exchange integrals and $R_{ln} = |\mathbf{R}_l - \mathbf{R}_n|$ is the distance between the l-th and n-th lattice sites. The first term in the Hamiltonian $H^{(s)}$ is the exchange energy between atoms of the basic material, and the summation is taken over those lattice sites which are occupied by atoms of the basic substance. The second term is the exchange energy between basic atoms and paramagnetic impurity atoms, with the summation over l taken, in this case, over those lattice sites occupied by basic atoms, and over n taken over those sites occupied by impurity atoms. The third term is the energy of the exchange interaction between paramagnetic impurity atoms, with the sum taken over sites occupied by these impurity atoms.

The exchange energy $H^{(s)}$ is not changed formally if there are diamagnetic impurities. We should bear in mind that the summation in (2) is already not taken over all lattice sites.

The energy of elastic vibrations of the lattice can be expressed, as far as terms quadratic in the atomic displacements, $\mathbf{u}_n$, from the equilibrium position, in the form

$$H^{(l)} = \tfrac{1}{2}\sum_{n} m_n\dot{\mathbf{u}}_n^2 + \tfrac{1}{2}\sum_{l,n} A_{ik}^{(1)}(R_{ln})\,u_l^i u_n^k + \sum_{l,n} A_{ik}^{(12)}(R_{ln})\,u_n^i u_l^k + \tfrac{1}{2}\sum_{l,n} A_{ik}^{(2)}(R_{ln})\,u_l^i u_n^k, \tag{3}$$

where m_n is the mass of the atom at the n-th site and A_{ik} are quantities characterizing the elastic interaction between atoms. The first term in the

Hamiltonian (3) represents the kinetic energy of the atoms, the second the elastic energy of atoms of the basic material, the third the elastic energy related to the interaction of the basic and impurity atoms and, finally, the fourth term represents the elastic energy of the impurity atoms. The summation in the first term of the Hamiltonian $H^{(l)}$ is taken over all lattice sites, in the second it is taken over those sites where there are atoms of the basic material, in the third term the summation over l is taken over those sites where there are atoms of the basic material, while over n it is taken over sites occupied by paramagnetic or diamagnetic impurities and in the fourth term the summation is taken over those sites where there are impurity atoms.

It is convenient to express the Hamiltonian H as the sum of a Hamiltonian H_0 describing free spin waves, phonons and the exchange energy of paramagnetic impurities and a Hamiltonian H_{int} characterizing the scattering of spin waves and phonons by impurities:

$$H = H_0 + H_{int},$$

$$H_0 = -\frac{1}{2}\sum_{l,n} J_1(R_{ln})\,\mathbf{s}_l\mathbf{s}_n + \frac{1}{2}m\sum_{ln}\dot{\mathbf{u}}_l^2 + \frac{1}{2}\sum_{l,n} A_{ik}^{(1)}(R_{ln})\,u_l^i u_n^k - \sum_{n,l} J_{12}(R_{ln})\,\mathbf{s}_l^0\sigma_n, \tag{4}$$

$$H_{int} = \sum_l\sum_n J_1(R_{ln})\,\mathbf{s}_l\mathbf{s}_n - \sum_l\sum_n J_{12}(R_{ln})(\sigma_n, \mathbf{s}_l - \mathbf{s}_l^0) + \sum_l\sum_n [A_{ik}^{(12)}(R_{ln}) - A_{ik}^{(1)}(R_{ln})]\,u_l^i u_n^k + \frac{1}{2}\sum_n \Delta m_n \dot{\mathbf{u}}_n^2. \tag{5}$$

Here m is the mass of a basic atom and $\Delta m_n = m - m_n$. In expression (5) for H_{int}, the summation over n is taken over the impurity sites. Besides impurities, we must also consider the existence of a rarer isotope of the basic substance. Corresponding to this we must also take the summation in the last term of the Hamiltonian H_{int} over sites occupied by atoms of the rarer isotope. We should point out that in writing the Hamiltonian Hint we have omitted terms proportional to the square of the impurity concentration.

Transforming from the operators $\mathbf{s}_l$ and u_l to creation operators $\overset{+}{a}_{\mathbf{k}}$ and annihilation operators $a_{\mathbf{k}}$ of the spin waves[3] and creation operators $\overset{+}{b}_{\mathbf{k}j}$ and annihilation operators $b_{\mathbf{k}j}$ of the phonons[4], we obtain

$$H_0 = \Sigma\varepsilon_{\mathbf{k}}\overset{+}{a}_{\mathbf{k}}a_{\mathbf{k}} + \Sigma\hbar\omega_{\mathbf{k}j}\overset{+}{b}_{\mathbf{k}j}b_{\mathbf{k}j}, \tag{6}$$

$$H_{int} = \Sigma\Phi_{\mathbf{k}_1\mathbf{k}_2}\overset{+}{a}_{\mathbf{k}_1}a_{\mathbf{k}_2} + \Sigma\Psi_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3j}\overset{+}{a}_{\mathbf{k}_1}a_{\mathbf{k}_2}b_{\mathbf{k}_3j} + \Sigma\chi_{\mathbf{k}_1j_1\mathbf{k}_2j_2}\overset{+}{b}_{\mathbf{k}_1j_1}b_{\mathbf{k}_2j_2} + \text{compl. conj.} \tag{7}$$

$\epsilon_{\mathbf{k}} = 2J_1 s\,(ak)^2$ is the energy of a spin wave with wave vector $\mathbf{k}$, a is the lattice constant, $\hbar\omega_{\mathbf{k}j} = \hbar c_j k$ is the energy of a phonon with wave vector $\mathbf{k}$ and polarization j and the summation is taken over all the vectors $\mathbf{k}$ and polarization j.

The quantities Φ, Ψ and χ are determined from the equations

$$\Phi_{\mathbf{k}_1\mathbf{k}_2} = \frac{1}{2}J_{12}\sigma f_1(\mathbf{k}_1, \mathbf{k}_2) + J_1 s a^2(\mathbf{k}_1, \mathbf{k}_2)\,[f_1(\mathbf{k}_1, \mathbf{k}_2) + f_2(\mathbf{k}_1, \mathbf{k}_2)], \tag{8}$$

$$\Psi_{\mathbf{k}_1\mathbf{k}_2,\,\mathbf{k}_3j} = 2i\sigma J_{12}(\hbar/2mN)^{1/2}\omega_{\mathbf{k}_3j}^{-1/2}\mathbf{e}_{\mathbf{k}_3j}\mathbf{k}_3 f_1(\mathbf{k}_2 + \mathbf{k}_3, \mathbf{k}_1), \tag{9}$$

$$\chi_{\mathbf{k}_1j_1,\,\mathbf{k}_2j_2} = -\frac{1}{4}\hbar\sqrt{\omega_{k_1j_1}\omega_{k_2j_2}}\,\{[m^{-1}\Delta m_1\mathbf{e}_{\mathbf{k}_1j_1}\mathbf{e}_{\mathbf{k}_2j_2} + \lambda_1(\mathbf{n}_1, \mathbf{n}_2)]\,f_1(\mathbf{k}_1, \mathbf{k}_2) + [m^{-1}\Delta m_2\mathbf{e}_{\mathbf{k}_1j_1}\mathbf{e}_{\mathbf{k}_2j_2} + \lambda_2(\mathbf{n}_1, \mathbf{n}_2)]\,f_2(\mathbf{k}_1, \mathbf{k}_2) + m^{-1}\Delta m_3 f_3(\mathbf{k}_1, \mathbf{k}_2)\}, \tag{10}$$

where J_1 and J_{12} are the exchange integrals for neighboring atoms; $\mathbf{e}_{\mathbf{k}j}$ is the unit polarization vector; $\lambda_1(\mathbf{n}_1, \mathbf{n}_2)$ and $\lambda_2(\mathbf{n}_1, \mathbf{n}_2)$ are dimensionless quantities of the order of unity, dependent on the directions of the wave vectors and polarizations of the phonons scattered by the impurities; Δm_1, Δm_2 and Δm_3 are respectively the differences between the masses of an atom of the basic material and of a paramagnetic impurity atom, a diamagnetic impurity atom and an atom of a rarer isotope. Finally, the function $f(\mathbf{k}_1, \mathbf{k}_2)$ describes the phase difference produced in the scattering of spin waves and phonons by impurities, and f_1, f_2 and f_3 refer, respectively, to scattering by paramagnetic and diamagnetic impurities and by atoms of a rarer isotope. The function $f(\mathbf{k}_1, \mathbf{k}_2)$ is determined by the formula

$$f(\mathbf{k}_1,\mathbf{k}_2) = \frac{1}{N}\sum_n \exp\{-i(\mathbf{k}_1 - \mathbf{k}_2, \mathbf{R}_n)\},$$

where the summation over n is taken over those lattice sites occupied by the corresponding impurity.

If we know the interaction Hamiltonian (6) we can calculate the thermal conductivity of a ferromagnetic dielectric determined by scattering of spin waves and phonons by impurities. For this purpose we write the kinetic equations, giving the change in the number of spin waves $n_{\mathbf{k}}$ and of phonons $N_{\mathbf{k}j}$ in the presence of a small temperature gradient:

$$\dot{n}_{\mathbf{k}}^{\mathrm{CT}} \equiv L_{\mathbf{k}}^{ss}\{n\} + L_{\mathbf{k}}^{sl}\{n, N\} = n_{\mathbf{k}}^0(n_{\mathbf{k}}^0 + 1)\,T^{-2}\varepsilon_{\mathbf{k}}\mathbf{v}_{\mathbf{k}}\nabla T, \tag{11}$$

$$\dot{N}_{\mathbf{k}j}^{\mathrm{CT}} \equiv L_{\mathbf{k}j}^{ll}\{N\} + L_{\mathbf{k}j}^{ls}\{n, N\} = N_{\mathbf{k}j}^0(N_{\mathbf{k}j}^0 + 1)\,T^{-2}\hbar\omega_{\mathbf{k}j}\mathbf{c}_{\mathbf{k}j}\nabla T,$$

where $\mathbf{V}_{\mathbf{k}} = \hbar^{-1}\partial\epsilon_{\mathbf{k}}/\partial\mathbf{k}$ is the velocity of a spin wave; $\mathbf{c}_{\mathbf{k}j}$ the velocity of a phonon of polarization j and wave vector $\mathbf{k}$; $\overset{0}{n}_{\mathbf{k}}$ and $\overset{0}{N}_{\mathbf{k}j}$ are the equilibrium Bose functions. The collision operator L is equal to:

$$L^{ss}_{\mathbf{k}_1}\{n\} = \frac{2\pi}{\hbar}\sum_{\mathbf{k}_2}|\Phi_{\mathbf{k}_1\mathbf{k}_2}|^2[(n_{\mathbf{k}_1}+1)n_{\mathbf{k}_2} - n_{\mathbf{k}_1}(n_{\mathbf{k}_2}+1)]\,\delta(\varepsilon_{\mathbf{k}_1}-\varepsilon_{\mathbf{k}_2}), \tag{12}$$

$$L^{sl}_{\mathbf{k}_1}\{n, N\} = \frac{2\pi}{\hbar}\sum_{\mathbf{k}_2\mathbf{k}_3 j}\{|\Psi_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3 j}|^2[(n_{\mathbf{k}_1}+1)n_{\mathbf{k}_2}N_{\mathbf{k}_3 j} - n_{\mathbf{k}_1}(n_{\mathbf{k}_2}+1)(N_{\mathbf{k}_3 j}+1)]\,\delta(\varepsilon_{\mathbf{k}_1}-\varepsilon_{\mathbf{k}_2}-\hbar\omega_{\mathbf{k}_3 j}) + |\Psi_{\mathbf{k}_2\mathbf{k}_1\mathbf{k}_3 j}|^2[(n_{\mathbf{k}_1}+1)(N_{\mathbf{k}_3 j}+1)n_{\mathbf{k}_2} - n_{\mathbf{k}_1}N_{\mathbf{k}_3 j}(n_{\mathbf{k}_2}+1)]\,\delta(\varepsilon_{\mathbf{k}_1}+\hbar\omega_{\mathbf{k}_3 j}-\varepsilon_{\mathbf{k}_2})\}, \tag{13}$$

$$L^{ll}_{\mathbf{k}j}\{N\} = \frac{2\pi}{\hbar}\sum_{\mathbf{k}_2 j_2}|\chi_{\mathbf{k}_1 j_1\mathbf{k}_2 j_2}|^2[(N_{\mathbf{k}_1 j_1}+1)N_{\mathbf{k}_2 j_2} - N_{\mathbf{k}_1 j_1}(N_{\mathbf{k}_2 j_2}+1)]\,\delta(\hbar\omega_{\mathbf{k}_1 j_1}-h\omega_{\mathbf{k}_2 j_2}), \tag{14}$$

$$L^{ls}_{\mathbf{k}j}\{N, n\} = \frac{2\pi}{\hbar}\sum_{\mathbf{k}_2\mathbf{k}_3}|\Psi_{\mathbf{k}_2\mathbf{k}_3\mathbf{k}_1 j}|^2[(N_{\mathbf{k}_1 j_1}+1)(n_{\mathbf{k}_2}+1)n_{\mathbf{k}_3} - N_{\mathbf{k}_1 j_1}n_{\mathbf{k}_2}(n_{\mathbf{k}_3}+1)]\,\delta(\hbar\omega_{\mathbf{k}_1 j_1}+\varepsilon_{\mathbf{k}_2}-\varepsilon_{\mathbf{k}_3}). \tag{15}$$

We shall look for solutions of (11) of the form

$$\begin{aligned} n_{\mathbf{k}} &= n^0_{\mathbf{k}} + n^0_{\mathbf{k}}(n^0_{\mathbf{k}}+1)T^{-2}\varepsilon_{\mathbf{k}}g(\varepsilon_{\mathbf{k}})(\mathbf{k}\nabla T),\\ N_{\mathbf{k}j} &= N^0_{\mathbf{k}j} + N^0_{\mathbf{k}j}(N^0_{\mathbf{k}j}+1)T^{-2}\hbar\omega_{\mathbf{k}j}f_j(\hbar\omega_{\mathbf{k}j})(\mathbf{k}\nabla T). \end{aligned} \tag{16}$$

Substituting (16) for the distribution functions of spin waves and phonons in the kinetic equations (11), and linearizing the collision operator for a small temperature gradient, we can obtain the following equations for the distribution functions g and f if we assume that $\epsilon_{\mathbf{k}}$ and $\omega_{\mathbf{k}}$ depend only on the magnitude of the wave vector **k**:

$$\begin{aligned} g(\varepsilon_{\mathbf{k}})[\delta L^{ss}_{\mathbf{k}}/\delta n_{\mathbf{k}} + \delta L^{sl}_{\mathbf{k}}/\delta n_{\mathbf{k}}] &= 1,\\ f_j(\hbar\omega_{\mathbf{k}j})[\delta L^{ll}_{\mathbf{k}j}/\delta N_{\mathbf{k}j} + \delta L^{ls}_{\mathbf{k}j}/\delta N_{\mathbf{k}j}] &= 1, \end{aligned} \tag{17}$$

where $\delta L_{\mathbf{k}}/\delta n_{\mathbf{k}}$ and $\delta L_{\mathbf{k}j}/\delta N_{\mathbf{k}j}$ are the variational derivatives of the collision operators with respect to the distribution functions, taken at $n_{\mathbf{k}} = n^0_{\mathbf{k}}$ and $N_{\mathbf{k}j} = N^0_{\mathbf{k}j}$.

From (17) we obtain

$$g^{-1}(\varepsilon_{\mathbf{k}_1}) = -\frac{2\pi}{\hbar}\sum_{\mathbf{k}_2}|\Phi_{\mathbf{k}_1\mathbf{k}_2}|^2\delta(\varepsilon_{\mathbf{k}_1}-\varepsilon_{\mathbf{k}_2}) - \frac{2\pi}{\hbar}\sum_{\mathbf{k}_2\mathbf{k}_3 j}\{|\Psi_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3 j}|^2(n_{\mathbf{k}_2}+N_{\mathbf{k}_3}+1)\,\delta(\varepsilon_{\mathbf{k}_1}-\varepsilon_{\mathbf{k}_2}-\hbar\omega_{\mathbf{k}_3 j}) - |\Psi_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3 j}|^2(n_{\mathbf{k}_2}-N_{\mathbf{k}_3 j})\,\delta(\varepsilon_{\mathbf{k}_1}+\hbar\omega_{\mathbf{k}_3 j}-\varepsilon_{\mathbf{k}_2})\}, \tag{18}$$

$$f^{-1}_{j_1}(\omega_{\mathbf{k}_1 j_1}) = -\frac{2\pi}{\hbar}\sum_{\mathbf{k}_2 j_2}|\chi_{\mathbf{k}_1 j_1,\,\mathbf{k}_2 j_2}|^2\delta(\hbar\omega_{\mathbf{k}_1 j_1}-\hbar\omega_{\mathbf{k}_2 j_2}) + \frac{2\pi}{\hbar}\sum_{\mathbf{k}_2\mathbf{k}_3}|\Psi_{\mathbf{k}_2\mathbf{k}_3\mathbf{k}_1 j}|^2(n_{\mathbf{k}_3}-n_{\mathbf{k}_2})\,\delta(\hbar\omega_{\mathbf{k}_1 j_1}+\varepsilon_{\mathbf{k}_2}-\varepsilon_{\mathbf{k}_3}). \tag{19}$$

We shall now determine the heat currents $\mathbf{S}_s$ and $\mathbf{S}_l$, carried by spin waves and phonons

$$\mathbf{S}_s = \sum_{\mathbf{k}}\varepsilon_{\mathbf{k}}\mathbf{v}_{\mathbf{k}}n_{\mathbf{k}}, \quad \mathbf{S}_l = \sum_{\mathbf{k}j}\hbar\omega_{\mathbf{k}j}\mathbf{c}_{\mathbf{k}j}N_{\mathbf{k}j}.$$

Using the distribution functions (16) and neglecting the dependence of the velocity of sound $\mathbf{c}_{\mathbf{k}j}$ on the direction of the wave vector **k**, we obtain[6]

$$\mathbf{S}_s = \frac{2}{3}\frac{V}{(2\pi)^3}\frac{T}{\hbar}\nabla T\int\left(\frac{\varepsilon}{T}\right)^3 n^0_{\mathbf{k}}(n^0_{\mathbf{k}}+1)g(\varepsilon_{\mathbf{k}})\,d\mathbf{k},$$

$$\mathbf{S}_l = \frac{1}{3}\frac{V}{(2\pi)^3}\frac{T}{\hbar}\nabla T\sum_j\left\{\int_{T/\hbar S_j}^{\infty}\left(\frac{\hbar\omega_{\mathbf{k}j}}{T}\right)^3 N^0_{\mathbf{k}j}(N^0_{\mathbf{k}j}+1)f_j(\hbar\omega_{\mathbf{k}j})\,d\mathbf{k} + \int_0^{T/S_j\hbar} f_j(T)\left(\frac{\hbar\omega_j}{T}\right)^2 N^0_{\mathbf{k}j}(N^0_{\mathbf{k}j}+1)\,d\mathbf{k}\right\}$$

From this the thermal conductivity

$$\varkappa = \varkappa_s + \varkappa_l,$$

where κ_s and κ_l are the thermal conductivities of spin waves and phonons:

$$\varkappa_s = -\frac{2}{3(2\pi)^3}\frac{T}{\hbar}\int\left(\frac{\varepsilon_{\mathbf{k}}}{T}\right)^3 n^0_{\mathbf{k}}(n^0_{\mathbf{k}}+1)g(\varepsilon_{\mathbf{k}})\,d\mathbf{k}, \tag{20}$$

$$\varkappa_l = -\frac{1}{3(2\pi)^3}\frac{T}{\hbar}\sum_j\left\{\int_{T/\hbar S_j}^{\infty}\left(\frac{\hbar\omega_{\mathbf{k}j}}{T}\right)^3 N^0_{\mathbf{k}j}(N^0_{\mathbf{k}j}+1)f_j(\hbar\omega_{\mathbf{k}j})\,d\mathbf{k} + \int_0^{T/S_j\hbar} f_j(T)\left(\frac{\hbar\omega_j}{T}\right)^2 N^0_{\mathbf{k}j}(N^0_{\mathbf{k}j}+1)\,d\mathbf{k}\right\}. \tag{21}$$

The functions $g(\epsilon_{\mathbf{k}})$ and $f_j(\hbar\omega_{\mathbf{k}j})$ which appear in (20) and (21) depend, in general, on the actual location of the impurity in the solid. If the impurity concentration is small enough so that $d\bar{k} \gg 1$ (d is the mean distance between the impurities and $\bar{k}$ is the mean value of the wave vector of a spin wave and phonon at a given temperature) we can neglect interference in the scattering of spin waves and phonons by separate impurities. This means that we can replace the term

$$|f(\mathbf{k}_1, \mathbf{k}_2)|^2 = \frac{1}{N^2}\Big|\sum_n \exp\{i(\mathbf{k}_1-\mathbf{k}_2, \mathbf{R}_n)\}\Big|^2$$

by ξ/N, where ξ are the mean concentrations of the respective impurities. In this limiting case expressions (20) and (21) for κ_s and κ_l take the form

$$\varkappa_s = \frac{4J_1 s}{3\pi\hbar a}\left(\frac{T}{2J_1 s}\right)^2\int_{\mu M_0/T}^{\infty}\frac{e^x x^3\,dx}{(e^x-1)^2\left[[\xi_p(J_{12}\sigma/2J_1 s)^2 + \frac{2}{3}(T/2J_1 s)^2 x^2\xi_d\right]}, \tag{22}$$

$$\varkappa_l = \frac{2\pi}{3}\sum_j\left\{\frac{\Theta_j^2}{\hbar aT}\int_1^{\infty}f_j(y)\,N^0_y(N^0_y+1)\,dy + f_j(1)\int_0^1 y^2N^0_y(N^0_y+1)\,dy\right\}, \tag{23}$$

where μ is the Bohr magneton, $M_0 = \mu/a^3$ is the magnetic moment per unit volume and

$$f_j^{-1}(y) = \left[\left(\frac{\Delta m_1}{m}\right)^2\overline{(\mathbf{e}_1\mathbf{e}_2)^2} + \overline{\lambda_1^2}\right]\xi_p + \left[\left(\frac{\Delta m_2}{m_0}\right)^2\overline{(\mathbf{e}_1\mathbf{e}_2)^2} + \overline{\lambda_2^2}\right]\xi_d + \xi_i\left(\frac{\Delta m_3}{m}\right)^2\overline{(\mathbf{e}_1\mathbf{e}_2)^2} + \xi_p\frac{2J_{12}\sigma}{mc_j^2}\left(\frac{\Theta_j}{2J_1 s}\right)^3\frac{2J_{12}\sigma}{T}\frac{(e^y-1)}{y^3}\times\int_0^{\infty}\frac{e^x\sqrt{x(x+y)}}{(e^x-1)(e^{x+y}-1)}\,dx.$$

ξ_p, ξ_d and ξ_i are the concentrations of paramagnetic and diamagnetic impurities and of atoms of a rarer isotope; the bar over the functions λ_1^2, λ_2^2, and $(\mathbf{e}_1\mathbf{e}_2)^2$ indicates integration over the angles, and $\Theta_j = \hbar c_j/a$.

In certain limiting cases, Eqs. (22) and (23) simplify considerably. We derive first expressions for the phonon conductivity.

If $\Theta_D \ll J_1$,

$$\varkappa_l \approx \Theta_D^2/\hbar a\xi T,$$

$$\xi = [(\Delta m_1/m)^2\,\overline{(\mathbf{e}_1\mathbf{e}_2)^2} + \overline{\lambda_1^2}]\,\xi_p + [(\Delta m_2/m)^2\,\overline{(\mathbf{e}_1\mathbf{e}_2)^2} + \overline{\lambda_2^2}]\,\xi_d + (\Delta m_3/m)^2\,\overline{(\mathbf{e}_1\mathbf{e}_2)^2}\,\xi_i. \tag{24}$$

If $\Theta_D \gg J_1$

$$\begin{aligned} \varkappa_l &\approx \Theta_D^2/\hbar a\xi T, & \xi_p &\ll (J_1/J_{12})^2(\xi_d+\xi_i)\,J_1T/\Theta_D^2; \\ \varkappa_l &\approx (J_1/J_{12})^2\,J_1/\hbar a\xi_p, & \xi_p &\gg (J_1/J_{12})^2(\xi_d+\xi_i)\,J_1T/\Theta_D^2. \end{aligned} \tag{25}$$

The expression for the conductivity given by equation (24) and by the first formula of (25) agrees with the phonon conductivity found earlier by Pomeranchuk[5] and by Klemens[6] from a consideration of the thermal conductivity of dielectrics.

The conductivity of the spin waves, κ_s is given by:

$$\begin{aligned} \varkappa_s &\approx (J_1/a\hbar\xi_p)(T/J_1)^2, & \xi_p &\gg (T/J_1)^2\,\xi_d, \\ \varkappa_s &\approx (J_1/a\hbar\xi_d)\ln(T/\mu M_0), & \xi_p &\ll (T/J_1)^2\,\xi_d. \end{aligned} \tag{26}$$

By using (24) — (26) it is easy to obtain expressions for the thermal conductivity of ferromagnetic dielectrics in various limiting cases.

We confine ourselves to the case when the impurity is either paramagnetic or diamagnetic. If the impurity is paramagnetic, then

$$\begin{aligned} \varkappa &\approx \Theta_D^2/\hbar a\xi_p T, & \Theta_D &\ll J_1; \\ \varkappa &\approx (J_1/J_{12})^2 J_1/a\hbar\xi_p, & \Theta_D &\gg J_1. \end{aligned} \tag{27}$$

If the impurity is diamagnetic, then

$$\begin{aligned} \varkappa &= (J_1/a\hbar\xi_d)\ln(T/\mu M_0), & \Theta_D &\gg T \gg \Theta_D^2/J_1; \\ \varkappa &= \Theta_D^2/a\hbar\xi_d T, & T &\ll \Theta_D^2/J_1. \end{aligned} \tag{28}$$

For paramagnetic impurities, therefore, the conductivity is determined by the scattering of phonons by impurities. For diamagnetic impurities, if $\Theta_D \gg T \gg \Theta_D^2/J_1$ the conductivity of a ferromagnetic dielectric is determined by scattering of spin waves by impurities, and for $T \ll \Theta_D^2/J_1$ by scattering of phonons by impurities.

According to the work of Akhiezer and one of the present authors,[7] the conductivity determined by umklapp processes is

$$\begin{aligned} \varkappa &\sim (mc^2/9\hbar)(C_l+2C_s)^2 a^5 \exp(\pi\Theta_D/T), & \Theta_D &\ll J_1; \\ \varkappa &\sim (T/a\hbar)\exp(\pi^2 J_1/T), & \Theta_D &\gg J_1. \end{aligned} \tag{29}$$

Here $C_s \sim a^{-3}(T/J_1)^{3/2}$ and $C_1 \sim a^{-3}(T/\Theta_D)^3$ are the heat capacities of the spin waves and phonons respectively. By comparing expressions (27), (28), and (29) it can be seen that collisions of spin waves and phonons with impurities are the dominant factors determining the conductivity if the impurity concentration is great enough to satisfy the inequality

$$\begin{aligned} \xi_p &\gg \frac{9\Theta_D^2}{mc^2a^6(C_l+2C_s)^2}\exp\left(-\frac{\pi\Theta_D}{T}\right) \text{ при } \Theta_D \ll J_1; \\ \xi_p &\gg \left(\frac{J_1}{J_{12}}\right)^2\frac{J_1}{T}\exp\left(-\frac{\pi^2 J_1}{2T}\right) \text{ при } \Theta_D \gg J_1 \end{aligned} \tag{30}$$

for the case of paramagnetic impurities, and the inequality

$$\begin{aligned} \xi_d &\gg \frac{J_1}{mc^2}\left(\frac{\Theta_D}{T}\right)^6 \ln\frac{T}{\mu M_0}\exp\left(-\frac{\pi\Theta_D}{T}\right) \text{ при } \Theta_D \gg T \gg \frac{\Theta_D^2}{J_1}; \\ \xi_d &\gg \left(\frac{\Theta_D}{T}\right)^2\exp\left(-\frac{\pi^2J_1}{2T}\right), \\ &\frac{\Theta_D^2}{mc^2T}\left(\frac{J_1}{T}\right)^3\exp\left(\frac{-\pi\Theta_D}{T}\right) \text{ при } T \ll J_1, \frac{\Theta_D^2}{J_1} \end{aligned} \tag{31}$$

for diamagnetic impurities.

We have not considered the scattering of spin waves and phonons on the boundaries of the specimen in calculating the conductivity. This is permissible if the mean free paths, l_l and l_s, of the phonons and spin waves are considerably smaller than the specimen dimensions L. The mean free paths can be derived from the conductivities κ_s and κ_l by using the relation which is well known in the kinetic theory of gases

$$\varkappa \approx vlC, \tag{32}$$

where C is the heat capacity, v the average velocity, and l the mean free path.

The mean velocity of spin waves is $v_s\,(T/J_1)^{1/2} \times J_1a/\hbar$, and the mean phonon velocity is the velocity of sound, c. Using expressions (24) — (26) for κ_l and κ_s, it is easy to see that the inequalities $l_s \ll L$ and $l_l \ll L$ hold if

$$\begin{aligned} \xi_p + (T/J_1)^2\,\xi_d &\gg a/L, \\ (J_{12}/\Theta_D)^2(T/J_1)^3\xi_p + (T/\Theta_D)^4(\xi_p+\xi_d+\xi_i) &\gg a/L. \end{aligned} \tag{33}$$

If the inequalities (33) are not satisfied the conductivity can be derived from the equation

$$\varkappa \sim (C_s v_s + C_l c)\,L. \tag{34}$$

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SOVIET PHYSICS JETP VOLUME 12, NUMBER 2 FEBRUARY, 1961

ON A CERTAIN MECHANISM OF ABSORPTION OF ULTRASOUND IN PARAMAGNETIC METALS IN A MAGNETIC FIELD

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One of the mechanisms of ultrasonic absorption in paramagnetic metals is discussed, namely, the appearance of currents as the result of variation of the temperature and the magnetic susceptibility. The magnitude of the effect is estimated.

If the susceptibility of a medium depends on temperature, and an external magnetic field is applied, then the magnetization varies along with the temperature as a longitudinal sound wave passes through each point in the medium. In a conducting medium, currents are induced in such a case, leading to an additional sound absorption. An estimate of the absorption of a plane wave in an isotropic medium, brought about by this effect, is carried out below.

It is evident that there will be no effect for $\mathbf{H} \parallel \mathbf{k}$ ($\mathbf{k}$ is the wave vector of the sound wave), since a longitudinal magnetization wave arises and no currents are induced. For $\mathbf{H} \perp \mathbf{k}$, the effect is at a maximum. In this case, as is easy to see from Maxwell's equations, a transverse current wave arises, propagating with the velocity of sound and polarized perpendicular to the vectors $\mathbf{H}$ and $\mathbf{k}$, with amplitude

$$j = -\sigma H\theta\left(\frac{\partial\mu}{\partial T}\right)\frac{v}{c}\left[1 + \frac{i}{2\pi^2}\left(\frac{\lambda}{\delta}\right)^2\right]^{-1}. \qquad (1)$$

Here σ is the conductivity, μ the magnetic permeability, v the speed of sound, λ the wavelength, δ the thickness of the skin layer, and θ the amplitude of the acoustic fluctuations of the temperature. The quantity θ is connected with the amplitude of the displacement in the sound wave, u. For adiabatic vibrations,[1]

$$|\theta| = akuT, \quad a = \alpha\rho\left(v_l^2 - \frac{4}{3}v_t^2\right) \Big/ C_p = \alpha E/3C_p(1 - 2\varkappa),$$

where α is the coefficient of thermal expansion, E is Young's modulus, and κ is the Poisson ratio.

The dissipation and the absorption coefficient are computed in the usual way with the aid of (1). The linear absorption coefficient is equal to

$$\gamma = \gamma_m\left[1 + \frac{1}{4\pi^4}\left(\frac{\lambda}{\delta}\right)^4\right]^{-1}, \qquad \gamma_m = \frac{\sigma}{2v\rho}\left(\frac{aHT}{c}\frac{\partial\mu}{\partial T}\right)^2. \qquad (2)$$

For low frequencies ($\lambda/\delta \gg 1$), we can neglect unity in the denominator, so that $\gamma = 4\pi^4(\delta/\lambda)^4\gamma_m$, i.e., the absorption coefficient increases in proportion to the square of the frequency. At high frequencies, when $\lambda/\delta \ll 1$ (which is true in practice even at $\omega \sim 10^8\,\text{sec}^{-1}$), a limiting value, equal to γ_m, is obtained.

The dependence of the absorption on the field and on the frequency is the same as takes place in the motion of a conducting medium in a sound wave in a magnetic field (as the result of eddy currents).[2] The absorption coefficient in the latter case is given by Eq. (2), except that of $aT\partial\mu/\delta T$ is replaced by μ. The ratio of the absorption coefficient found from (2) to the absorption coefficient due to the eddy currents is thus equal to $(aT/\mu)^2(\partial\mu/\partial T)^2$. This ratio is almost always small. Exceptions are the strongly paramagnetic rare-earth metals close to the ferro- and antiferromagnetic transition points, where the derivative $\partial\mu/\partial T$ is relatively large. Thanks to this situation, the effect considered can be shown to be even stronger than the effect brought about by the eddy currents. In dysprosium, for example, for T = 180°K (close to the Néel temperature of 175°K) we have $\partial\mu/\partial T \approx 0.01\,\text{deg}^{-1}$.[3] We did not succeed in obtaining experimental data which permitted the calculation of the parameter a for rare-earth metals. For "tabular" metals, a is of the order of unity: aluminum — 2.3; copper — 1.8; silver — 2.5; and platinum — 2.3. Therefore, setting $a \sim 1$, we find $(aT/\mu)^2(\partial\mu/\partial T)^2 \approx 3$. For $\sigma \sim 10^{16}\,\text{sec}^{-1}$ (reference 4), $v \approx 3 \times 10^5$ cm/sec, $\rho = 8.6$ g/cm^3 and $H \sim 10^4$ oe, the absorption coefficient (2) is of the order $\gamma_m \sim 10^{-3}\,\text{cm}^{-1}$.

We note that relaxation of the magnetization has not been taken into account in the estimate given above. This is obviously permissible under the condition $\omega\tau \ll 1$, where τ is the time of spin-lattice relaxation. As S. A. Al'tshuler kindly informed the author, there is a basis for expecting the time of spin-lattice relaxation in rare earth metals to be very small, and therefore the condition $\omega\tau \ll 1$ can be assumed to be satisfied up to very high frequencies.

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INELASTIC FINAL-STATE INTERACTIONS AND NEAR-THRESHOLD SINGULARITIES

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It is shown that non-monotonic energy variations can occur in the energy spectrum of particle a from a reaction of the type $A + B \rightarrow a + C + D$ in the neighborhood of the threshold for the reaction $C + D \rightarrow E + F$. As an example, we analyze the spectrum of K mesons from the reaction $N + N \rightarrow \Lambda + N + K$ in the region of the energy of the $\Lambda - N$ pair close to the threshold for the process $\Lambda + N \rightarrow \Sigma + N$. For the process $p + p \rightarrow \Lambda + N + K$ we find the energy spectrum of the K mesons when the incident nucleons are unpolarized, and the polarization of the baryons when the incident nucleons are polarized.

We discuss the non-monotonic energy variations in the spectra of particles for some other reactions. In the Appendix we analyze the production of $Y-K$ pairs in np collisions and discuss the case of a scalar K particle.

1. INTRODUCTION

It is known that in processes of production of particles an interaction between two of the particles formed affects the energy spectrum and angular distribution of the third particle. In certain cases, the effect of final-state interaction can be separated from the primary mechanism for production of the particles. This occurs when the effective radius for the primary interaction is much less than the radius of interaction of a pair of particles in the final state. In addition, if the interaction of the pair of particles with other emerging particles is weak, the interaction of the two particles in the final state can be characterized by a two-particle scattering length.

The theory of final-state interaction was applied by Migdal,[1] Brueckner and Watson,[2] and Paruntseva[3] to meson production in NN collisions. Recently Henley[4] and Feldman and Matthews[5] applied it to the analysis of the reaction

$$N + N \rightarrow Y + N + K. \qquad (1)$$

They showed that the energy spectrum of the K mesons is strongly distorted by the effect of the YN interaction.

Karplus and Rodberg[6] generalized the theory of final-state interaction to the case where the strong interaction in the final state can lead to an inelastic process.

In the present paper we shall show that in the neighborhood of the threshold for production of the Σ hyperon certain anomalies occur in the energy spectrum of the K particles formed together with the Λ hyperons. They are a new example of near-threshold anomalies which have been extensively studied in recent years.[7]

In addition to the cross section for the new inelastic process, the shape and appearance of near-threshold anomalies depend on the spin and parity of the particles. The study of these anomalies with sufficient accuracy can enable us to determine properties of the produced particles. On the assumption that the final state of reaction (1) is described by singlet and triplet s waves of the Y-N system, we analyze in the second section of the present paper the kinematics of the reaction and obtain expressions for the energy spectrum of the K mesons and the polarization of the Λ particles and nucleons when the incident beam of nucleons is polarized.

In the third section, starting from the unitarity of the S matrix and the analyticity of the reaction amplitude, we give a general formulation of the theory of inelastic final-state interactions.

In Sec. 4 we consider local near-threshold anomalies in the energy spectrum of K mesons in the reaction $N + N \rightarrow \Lambda + N + K$ in the neighborhood of the threshold for formation of the Σ hyperon.

In conclusion, we mention some other similar processes and discuss the possible generalization of the method developed here to these processes.

2. KINEMATICS. PHENOMENOLOGICAL ANALYSIS.

We introduce Jacobi coordinates in the final state of the three-particle system:

$$\mathbf{R} = \frac{M_N\mathbf{r}_N + M_Y\mathbf{r}_Y + M_K\mathbf{r}_K}{M_N + M_Y + M_K}, \qquad \boldsymbol{\rho} = \mathbf{r}_K - \frac{M_N\mathbf{r}_N + M_Y\mathbf{r}_Y}{M_N + M_Y},$$
$$\mathbf{r} = \mathbf{r}_N - \mathbf{r}_Y, \tag{2}$$

where M_N, M_Y and M_K are respectively the masses of the nucleon, hyperon and K meson; $\mathbf{r}_N$, $\mathbf{r}_Y$ and $\mathbf{r}_K$ are their coordinates. The momenta conjugate to $\mathbf{R}$, $\boldsymbol{\rho}$, and $\mathbf{r}$ will be $\mathbf{p}_R$, $\mathbf{p}_Y$ and $\mathbf{q}$ respectively. The total energy E in the new variables is equal to (c.m.s.)

$$E = p_Y^2/2m_Y + q^2/2\mu + M_K + M_Y - M_N; \tag{3}$$

$$M = M_N + M_Y + M_K, \qquad m_Y = \frac{M_N M_Y}{M_N + M_Y},$$
$$\mu = \frac{M_K(M_N + M_Y)}{M_K + M_N + M_Y}. \tag{4}$$

The phase volume of the final state is expressed in terms of p_Y and q as follows:

$$dJ = m_Y p_Y \, d\Omega_Y \, q^2 \, dq \, d\Omega_q, \tag{5}$$

where $d\Omega_Y$ and $d\Omega_q$ are the solid angles for the momenta $\mathbf{p}_Y$ and $\mathbf{q}$ respectively.

To be specific, we consider the reaction

$$p + p \rightarrow \Lambda + p + K^+ \tag{6}$$

below the threshold of the reaction

$$p + p \rightarrow \Sigma^0 + p + K^+. \tag{7}$$

The admissible energy for the final state of reaction (6) in the c.m.s. does not exceed 80 Mev, so that we may assume that the particles which are formed are in an s state.

Let us represent the S matrix element in the form

$$\langle \Lambda p K^+ | S | pp \rangle = -2\pi i \delta(E_i - E_f) \langle \Lambda p K^+ | T | pp \rangle. \tag{8}$$

If the K meson is a pseudoscalar particle, the spin structure of the T matrix has the form

$$\langle \Lambda p K | T | pp \rangle = A_\Lambda (\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2, \mathbf{k})$$
$$+ B_\Lambda \{(\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2, \mathbf{k}) + i([\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2]\mathbf{k})\}$$
$$+ C_\Lambda \{(\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2, \mathbf{k}) - i([\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2]\mathbf{k})\}, \tag{9}$$

where $\boldsymbol{\sigma}$ is the spin matrix, $\mathbf{k}$ is a unit vector along the direction of the incident proton: A_Λ, B_Λ, and C_Λ are scalar functions of the total energy E and the relative momentum P_Λ of the $\Lambda-N$ pair. Since there are two identical particles in the initial state, the elements of the T matrix must be antisymmetrized with respect to the two initial protons. It can be shown that this results in $B_\Lambda = 0$.

The expression for the cross section for reaction (6) with unpolarized particles has the form

$$\frac{d\sigma}{d\Omega_\Lambda \, d\Omega_q \, dT} = (2\pi)^4 \frac{E}{2(E^2 - 4M_N^2)^{1/2}}$$
$$\times (2m_\Lambda\mu)^{3/2} [T(T_{max} - T)]^{1/2}$$
$$\times [|A_\Lambda + C_\Lambda|^2 + |A_\Lambda - C_\Lambda|^2 + 2|C_\Lambda|^2], \tag{10}$$

where $T = q^2/2\mu$ is the kinetic energy of the K meson with respect to the center of mass of the $\Lambda-N$ system.

If the protons in the initial state are polarized (with polarization vector $\mathbf{P}$), the polarization vector of the Λ particle in the final state, $\mathbf{P}_\Lambda$, will be

$$\mathbf{P}_\Lambda [|A_\Lambda + C_\Lambda|^2 + |A_\Lambda - C_\Lambda|^2 + 2|C_\Lambda|^2] = 2[|A_\Lambda - C_\Lambda|^2$$
$$- |C_\Lambda|^2](\mathbf{kP})\mathbf{k} + [|A_\Lambda - C_\Lambda|^2 - |A_\Lambda + C_\Lambda|^2]\mathbf{P}. \tag{11}$$

The expression for the polarization of the nucleon in the final state differs from (11) by the sign in front of C_Λ.

3. ELASTIC FINAL-STATE INTERACTION

Let us look at the unitarity condition

$$\langle \Lambda p K | T - T^+ | pp \rangle$$
$$= 2\pi i \sum_n \langle \Lambda p K | T | n \rangle \langle n | T^+ | pp \rangle \delta(E_i - E_n), \tag{12}$$

where $|n\rangle$ is a possible intermediate state lying on the same energy surface as the initial state. Let us assume that in the region of energy considered the imaginary part of the T matrix is related mainly to strong interaction in the $\Lambda-p$ system. Then we may neglect on the right side of (12) all intermediate states except for ΛpK states, and approximately replace $\langle \Lambda pK | T | \Lambda' p' K' \rangle$ by $\langle \Lambda p | T | \Lambda' p' \rangle \langle K | K' \rangle$. This means that we are neglecting the interaction between the K meson and the $\Lambda-p$ pair.

In the low energy region the matrix element $\langle \Lambda p | T | \Lambda' p' \rangle$ is equal to

$$\langle \Lambda N | T | \Lambda' N' \rangle$$
$$= (4\pi^2 p_\Lambda m_\Lambda)^{-1} \left[\tfrac{1}{4}(3 + \boldsymbol{\sigma}_1\boldsymbol{\sigma}_2)\alpha_3 + \tfrac{1}{4}(1 - \boldsymbol{\sigma}_1\boldsymbol{\sigma}_2)\alpha_1\right], \tag{13}$$

where

$$\alpha_3 = e^{i\delta_3}\sin\delta_3, \qquad \alpha_1 = e^{i\delta_1}\sin\delta_1, \tag{14}$$

and δ_1 and δ_3 are the scattering phases in the singlet and triplet states respectively.

Using all these assumptions and taking account of invariance under time reversal, we find from (12)

$$\operatorname{Im} A_\Lambda = \frac{\operatorname{Re} \alpha_3}{1 - \operatorname{Im} \alpha_3} \operatorname{Re} A_\Lambda = \frac{\operatorname{Im} \alpha_3}{\operatorname{Re} \alpha_3} \operatorname{Re} A_\Lambda = \tan \delta_3 \operatorname{Re} A_\Lambda ,$$

$$\operatorname{Im} C_\Lambda = \tan \delta_1 \operatorname{Re} C_\Lambda ,$$

$$A_\Lambda = (1 + i \tan \delta_3) \operatorname{Re} A_\Lambda \approx (1 + i a_3 p_\Lambda) \operatorname{Re} A_\Lambda ,$$

$$C_\Lambda = (1 + i \tan \delta_1) \operatorname{Re} C_\Lambda \approx (1 + i a_1 p_\Lambda) \operatorname{Re} C_\Lambda . \qquad (15)$$

From (15) we see that for $\delta \rightarrow 0$, i.e., in the absence of final-state interaction, the quantities A_Λ and C_Λ are real functions.

In the energy region we are considering, the matrix elements of the reaction matrix are functions of two quantities: E — the total energy, and ω — the total energy of the $\Lambda - p$ system. If all the singularities of the amplitude are associated with physical processes, then A_Λ and C_Λ as analytic functions of ω and E are representable in the form

$$(p_\Lambda a)^{-1} e^{i\delta(\omega)} \sin \delta(\omega) f(\omega) F_\Lambda(E), \qquad (16)$$

where $f(\omega)$ is an entire function which, for small values of the energy, can be replaced by a constant.

Thus we finally approximate A_Λ and C_Λ by expressions

$$A_\Lambda = \frac{1}{p_\Lambda a_3} e^{i\delta_3} \sin \delta_3 \cdot A_\Lambda^0 , \qquad C_\Lambda = \frac{1}{p_\Lambda a_1} e^{i\delta_1} \sin \delta_1 \cdot C_\Lambda^0 , \qquad (16')$$

where a_3 and a_1 are the triplet and singlet Λp-scattering lengths in the s state, while A_Λ^0 and C_Λ^0 can be regarded approximately as real functions of the total energy E alone. Consequently, taking account of the unitarity of the S matrix and the analyticity of the reaction amplitude leads directly to the main result of the theory of final-state interaction (cf., for example, the paper of Gribov[8]).

By using (16) the expressions for the reaction cross section and the polarization of the Λ particles can be represented as

$$\frac{d\sigma}{dT} = (2\pi)^4 \frac{E}{2(E^2 - 4M_N^2)^{1/2}} (4\pi)^2 (2m_\Lambda \mu)^{3/2} [T(T_{max} - T)]^{1/2}$$

$$\times \left[2 \frac{\sin^2 \delta_3}{(p_\Lambda a_3)^2} |A_\Lambda^0|^2 + 4 \frac{\sin^2 \delta_1}{(p_\Lambda a_1)^2} |C_\Lambda^0|^2 \right], \qquad (17)$$

$$\mathbf{P}_\Lambda \left[\frac{\sin^2 \delta_3}{(p_\Lambda a_3)^2} |A_\Lambda^0|^2 + 2 \frac{\sin^2 \delta_1}{(p_\Lambda a_1)^2} |C_\Lambda^0|^2 \right] = \left[\frac{\sin^2 \delta_3}{(p_\Lambda a_3)^2} |A_\Lambda^0|^2 \right.$$

$$\left. + 2 A_\Lambda^0 C_\Lambda^0 \frac{\sin \delta_1 \sin \delta_3 \cos(\delta_1 - \delta_3)}{p_\Lambda^2 a_3 a_1} \right] (\mathbf{kP}) \mathbf{k}$$

$$- 2 A_\Lambda^0 C_\Lambda^0 \frac{\sin \delta_1 \sin \delta_3 \cos(\delta_1 - \delta_3)}{p_\Lambda^2 a_3 a_1} \mathbf{P} \qquad (18)$$

$p_\Lambda^2 = 2m_\Lambda (T_{max} - T)$. If we change the sign in front of C_Λ^0 on the right of equation (18), we obtain the expression for the polarization of the recoil nucleons. Expressions (17) and (10) can be considered as a generalization of the results of Henley, who neglected the dependence of the reaction matrix on spin.

From (17) and (18) we see that the investigation of the energy spectrum of K mesons and, in particular, of the polarization of Λ particles and nucleons is very important for the determination of the Λp-scattering lengths.

4. INELASTIC INTERACTION. NEAR-THRESHOLD SINGULARITIES.

As the energy is increased, the Σ channel is opened, and we may expect a change in the spectrum of K mesons and other quantities for the ΛKp channel. In this case, in the unitarity condition (8), we must consider as a possible intermediate state the state $|\Sigma NK>$. We shall restrict ourselves to interaction in s-states.

As in the preceding section we assume that*

$$\langle \Lambda NK | T | \Sigma N'K' \rangle \approx \langle \Lambda N | T | \Sigma N' \rangle \langle K | K' \rangle ,$$

and use the fact that

$$\langle \Lambda N | T | \Sigma N \rangle = [4\pi^2 p_\Lambda^{1/2} p_\Sigma^{1/2} m_\Lambda^{1/2} m_\Sigma^{1/2}]^{-1}$$

$$\times \left[\frac{1}{4} (3 + \sigma_1 \sigma_2) \beta_3 + \frac{1}{4} (1 - \sigma_1 \sigma_2) \beta_1 \right] . \qquad (19)$$

where the indices Λ and Σ denote quantities in the corresponding channels, while

$$p_\Sigma = [2m_\Sigma (E' - T)]^{1/2}, \qquad E' = E - M_\Sigma + M_\Lambda . \qquad (20)$$

Assuming that there are no bound states of the p-Σ system, we represent the energy dependence of β_3 and β_1 in the low-energy region in the form

$$\beta_3 = b_3 p_\Sigma^{1/2} , \qquad \beta_1 = b_1 p_\Sigma^{1/2} , \qquad (21)$$

if the internal parities of Σ and Λ are the same.

The influence of the Σ channel shows itself for the Λ channel not only as an additional term in the unitarity condition (8), but also as an additional term in the matrix element of the Λp scattering matrix proportional to p_Σ:

$$\alpha_3 = \alpha_3^0 + i c_3 p_\Sigma , \qquad \alpha_1 = \alpha_1^0 + i c_1 p_\Sigma , \qquad (22)$$

where

$$c_{1,3} = (p_\Lambda / 4\pi) \sigma_{1,3}^{\Sigma,\Lambda} , \qquad (23)$$

and $\sigma_{2j}^{\Sigma,\Lambda}$ is the total cross section for the reaction $\Sigma + N \rightarrow \Lambda + N$ in the state with angular momentum j.

Using (19)-(23), we find from (8)

$$\operatorname{Im} C_\Lambda = (\operatorname{Im} C_\Lambda)_{p_\Sigma = 0} + C'_\Lambda p_\Sigma ,$$

$$\operatorname{Im} A_\Lambda = (\operatorname{Im} A_\Lambda)_{p_\Sigma = 0} + A'_\Lambda p_\Sigma , \qquad (24)$$

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*The inclusion of terms of the type $<pp|T^+|pp>$ $<pp|T|YN'K'>$, which are small for this reaction, but are necessary in other cases, complicates the expressions but does not change the fundamental result.

where $(\delta \neq \pi/2)$

$$A'_{\Lambda} = A^0_{\Lambda}(p_{\Lambda}/4\pi)\,\sigma_3^{\Sigma,\Lambda}(p_{\Sigma}=0)\ \tan^2\delta_3 + A^0_{\Sigma} b_3/\cos^2\delta_3,$$
$$C'_{\Lambda} = C^0_{\Lambda}(p_{\Lambda}/4\pi)\,\sigma_1^{\Sigma,\Lambda}(p_{\Sigma}=0)\ \tan^2\delta_1 + C^0_{\Sigma} b_1/\cos^2\delta_1. \tag{25}$$

The relation (24) is valid when the kinetic energy T of the K meson is less than E′. For $T > E'$ the production of a real Σ particle becomes impossible, and we must replace p_{Σ} by ik_{Σ}, where $k_{\Sigma} = \sqrt{2m_{\Sigma}(T - E')}$, $T > E'$, so that the term which depends linearly on k_{Σ} appears in the real part of the reaction amplitude.

The presence of terms proportional to $p_{\Sigma}(T < E')$ and $k_{\Sigma}(T > E')$ causes the derivative with respect to the energy to become infinite both in the energy spectrum of the K mesons and in the energy dependence of the polarization of Λ particles (and nucleons). The order of magnitude of these anomalies is given by (24) and (25), and their shape depends on the relative sign of A^0_{Λ}, A^0_{Σ}, $b_{3,1}$ and δ. All four cases of anomalies which have been discussed in the literature for binary reactions can also occur in this present case.

All of the expressions in Secs. 2, 3, and 4 were given for the production of particles in pp collisions. It is not difficult to generalize them to the case of np collisions. This is done in the Appendix. We also discuss there the case of a scalar K particle.

We note that, in the general case also, the quantities which replace A'_{Λ} and C'_{Λ} have terms which are directly related to the final-state interaction, as well as terms which are not caused by it.

We emphasize that the expressions obtained in the present section refer to interaction in an s state of the final system. The relatively large mass difference of the Λ and Σ hyperons makes it difficult to apply the theory of inelastic interaction to the analysis of reaction (1), but this does not change the basic assertion that there is a non-monotonic behavior in the spectrum and the causes for its occurrence.

It was shown earlier[9] that the direct analytic continuation $p_{\Sigma} \rightarrow ik_{\Sigma}$ can not be carried out when there is a resonance in the neighborhood of the threshold. In this case, it is necessary to make use of dispersion relations. Since the analytic behavior of the reaction amplitude as a function of ω is not known, we have not carried out such an analysis. However, even if such a resonance occurs, we may expect non-monotonic variation with energy for a relative energy of the Λ–N pair equal to the threshold for the new channel.

If Σ and Λ have opposite parities, the first term of the expansion in (22) starts with p^3_{Σ} and only the second derivative with respect to the energy becomes infinite. Consequently, the study of threshold anomalies in the energy spectrum of K mesons with sufficiently high accuracy may prove important for determining the relative parity of the Σ and Λ particles.

5. DISCUSSION

Thus, endothermic inelastic interactions of the type C + D → E + F in the final state of the reaction A + B → a + C + D can give rise to non-monotonic variations with energy in the spectrum of the particles a, whose form can be determined from the condition of analyticity and unitarity of the S matrix. To investigate these singularities experimentally requires, of course, good accuracy and high energy resolution, but as a result of discovering them and studying them one can obtain information concerning the interaction of unstable particles, their spins and parities.

Earlier we have treated the production of hyperons and K mesons in NN collisions. We mention various other processes in which similar anomalies can occur whose study may give information concerning the interaction of unstable particles.

In the spectrum of π^+ mesons from the reaction

$$K^- + p \rightarrow \Lambda + \pi^- + \pi^+ \tag{26}$$

in the neighborhood of the threshold for

$$\pi^- + \Lambda \rightarrow \Sigma^- + \pi^0 \tag{27}$$

there will occur an anomaly whose magnitude and character will be related to K^-p scattering at low energies via the reaction amplitude (27).

In the spectrum of protons from the process for production of π mesons by K mesons

$$K^- + p \rightarrow p + \pi^0 + K^- \tag{28}$$

an anomaly may occur for an energy corresponding to the threshold for the reaction

$$\pi^0 + K^- \rightarrow \overline{K}^0 + \pi^-, \tag{29}$$

if there exist forces leading to such a reaction.

If one attempts to construct a Lagrangian for the πK interaction and does not consider interactions containing derivatives, the expression obtained

$$L_{int} = g\,(\varphi^i_\pi\cdot\varphi^i_\pi)\,(\varphi^k_\Theta\cdot\varphi^k_\Theta)$$

is invariant with respect to rotation of the isotopic spin of each of the particles, and all processes for K π scattering with charge exchange are forbidden. Under more general assumptions one does not obtain a forbiddenness for reaction (20), so that the observation of a non-monotonic variation with energy in the proton spectrum from reaction (28) would be of interest from the point of view of the study of the symmetry of the π K interaction.

Among reactions in which two π mesons participate, it is interesting to note that in the distribution of nucleons from the reactions

$$\gamma + p \to p + \pi^0 + \pi^0, \qquad \pi^- + p \to n + \pi^0 + \pi^0 \qquad (30)$$

there may occur similar anomalies for a relative energy of the π^0 mesons exceeding 9 Mev, where the charge exchange reaction

$$\pi^0 + \pi^0 \to \pi^- + \pi^+ \qquad (31)$$

becomes possible. Including threshold phenomena in the reaction (31) can have significant effects in the theory of $\pi\pi$ interaction at low energies.

The existence of a threshold in reaction (31) can lead to a non-monotonicity in the spectrum of charged π mesons from τ' decay.

$$\tau^\pm \to \pi^\pm + \pi^0 + \pi^0.$$

Analogously to the reactions (30), in the spectrum of nucleons from the reactions

$$\gamma + p \to p + K^- + K^+, \qquad \pi^- + p \to n + K^+ + K^- \qquad (32)$$

in the neighborhood of the threshold for the reaction

$$K^+ + K^- \to \overline{K}{}^0 + K^0 \qquad (33)$$

there may occur energy anomalies associated with KK interaction. Moreover, in the final state of reaction (33), there is no Coulomb interaction which might mask the non-monotonicity (cf. the paper of Newton and Fonda[10]).

In the spectrum of protons from the reaction

$$p + p \to p + p + \pi^0 \qquad (34)$$

near the threshold for

$$\pi^0 + p \to n + \pi^+$$

and in the spectrum of π^+ mesons from the reaction

$$p + p \to n + p + \pi^+ \qquad (35)$$

near the threshold for

$$n + p \to d + \pi^0$$

there will also be energy non-monotonicities.*

APPENDIX

A. PRODUCTION OF A PSEUDOSCALAR K MESON IN np COLLISION

In np collisions there are two possibilities for production of Λ particles

$$n + p \to \Lambda + p + K^0, \qquad (A.1)$$

$$n + p \to \Lambda + n + K^+. \qquad (A.2)$$

We denote the reaction amplitudes in the singlet and triplet isotopic spin states by T_0 and T_1 respectively. The reaction (A.1) is then described by the amplitude $\frac{1}{2}(T_0 + T_1)$, and the reaction (A.2) by the amplitude $\frac{1}{2}(T_1 - T_0)$. The spin dependence of the isotopic triplet amplitude is given by (9) with $B_\Lambda = 0$, while

$$\langle \Lambda NK \,|\, T_0 \,|\, NN\rangle = B_\Lambda\,\{(\sigma_1 - \sigma_2,\ \mathbf{k}) + i\mathbf{k}\,[\sigma_1\times\sigma_2]\}. \qquad (A.3)$$

Under the assumptions made earlier we can take account of final-state interaction by setting

$$B_\Lambda = B^0_\Lambda\,(p_\Lambda a_3)^{-1}\,e^{i\delta_3}\sin\delta_3, \qquad (A.4)$$

where B^0_Λ is a function of the total energy E.

The expressions for the cross sections for production and polarization of the Λ particles have the following forms:

$$d\sigma = (2\pi)^4\,\frac{E}{2\,[E^2 - 4M_N^2]^{1/2}}\,\frac{1}{4}\,(4\pi)^2\,(2m_\Lambda\mu)^{3/2}\,[T\,(T_{max} - T)]^{1/2}\,dT$$
$$\times\,[\,|A_\Lambda + C_\Lambda \pm B_\Lambda|^2 + |A_\Lambda - C_\Lambda \mp B_\Lambda|^2 + 2\,|C_\Lambda \mp B_\Lambda|^2], \qquad (A.5)$$

$$\mathbf{P}_\Lambda\,[\,|A_\Lambda + C_\Lambda \pm B_\Lambda|^2 + |A_\Lambda - C_\Lambda \mp B_\Lambda|^2 + 2\,|C_\Lambda \mp B_\Lambda|^2]$$
$$= 2\,[\,|A_\Lambda + C_\Lambda \pm B_\Lambda|^2 - |C_\Lambda \mp B_\Lambda|^2]\,(\mathbf{kP})\,\mathbf{k}$$
$$+ [\,|A_\Lambda - C_\Lambda \mp B_\Lambda|^2 - |A_\Lambda + C_\Lambda \pm B_\Lambda|^2]\,\mathbf{P}. \qquad (A.6)$$

The plus sign in front of B^0_Λ holds for reaction (A.1), and the minus sign for (A.2). From (A.5) and (A.6) it is easy to obtain the "intensity rules":

$$d\sigma\,(np \to \Lambda pK^0) = d\sigma\,(np \to \Lambda nK^+), \qquad (A.7)$$

$$\mathbf{P}_\Lambda\,(np \to \Lambda pK^0) = \mathbf{P}_\Lambda\,(np \to \Lambda nK^+) \quad \text{for } \mathbf{P}\,\|\,\mathbf{k}. \qquad (A.8)$$

These relations are obtained on the assumption

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*The scattering lengths for low energies of the π^0-p system differ from those obtained on the assumption of isotopic invariance because of the presence of non-monotonicities which violate isotopic invariance and are related to the reaction

$$\pi^0 + p \to n + \pi^+.$$

An estimate using dispersion relations gives a correction ~ 5%.

that one need only consider the s wave in the final state. They can be used for an experimental check of this assumption.

B. PRODUCTION OF A SCALAR K MESON IN NN COLLISIONS

In this case

$$\langle \Lambda NK \,|\, T_1 \,|\, NN \rangle = A_\Lambda, \quad \langle \Lambda NK \,|\, T_0 \,|\, NN \rangle = B_\Lambda\,(\boldsymbol{\sigma}_1 \boldsymbol{\sigma}_2); \tag{B.1}$$

$$A_\Lambda = A^0_\Lambda\,(p_\Lambda a_1)^{-1}\, e^{i\delta_1} \sin\delta_1, \quad B_\Lambda = B^0_\Lambda\,(p_\Lambda a_3)^{-1}\, e^{i\delta_3} \sin\delta_3. \tag{B.2}$$

If we introduce

$$f\,(NN \rightarrow \Lambda NK) = \frac{d\sigma\,(NN \rightarrow \Lambda NK) \cdot 8\,(E^2 - 4M_N^2)^{1/2}}{dT\;[T\,(T_{max} - T)]^{1/2}\,(2\pi)^4\, E\,(4\pi)^2\,(2m_\Lambda\,\mu)^{3/2}},$$

the cross section and polarization of the Λ particles in all three reactions are given by

$$f\,(pp \rightarrow \Lambda pK^+) = |\, A^0_\Lambda\,|^2\,(p_\Lambda\, a_1)^{-2} \sin^2\delta_1,$$

$$\mathbf{P}_\Lambda(pp \rightarrow \Lambda pK) = \mathbf{P};$$

$$f\,(np \rightarrow \Lambda NK) = f\,(pp \rightarrow \Lambda pK^+) + 3\,|\,B_0\,|^2\,(p_\Lambda\, a_3)^{-2} \sin^2\delta_3, \tag{B.3}$$

$$\mathbf{P}_\Lambda\,(np \rightarrow \Lambda NK) = (\,|\,A_\Lambda|^2 - |\,B_\Lambda|^2)\,(\,|\,A_\Lambda|^2 + 3\,|\,B_\Lambda|^{\,2})^{-1}\,\mathbf{P}. \tag{B.4}$$

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CHANGE OF THE ADIABATIC INVARIANT OF A PARTICLE IN A MAGNETIC FIELD. I

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The change of the adiabatic invariant is found for a particle moving in an axially symmetrical inhomogeneous magnetic field. The problem is solved for the usual model Hamiltonian (1).

THE problem of the conservation of adiabatic invariants, and in particular of the magnetic moment of a charged particle in a magnetic field, has recently been dealt with in a number of papers;[1,2] these papers have shown that the change of the adiabatic invariant is less than any power of the adiabatic parameter. In reference 3 the change of the adiabatic invariant was calculated for an oscillator whose frequency depends on the time. It is obvious that the change of the adiabatic invariant must be the same for a particle in a time-varying uniform magnetic field, under the condition $H \rightarrow \text{const}$ as $t \rightarrow \pm\infty$. A more complicated problem is to calculate the change of the adiabatic invariant in a magnetic field that varies in space. The solution of this problem is the purpose of the present paper.

1. STATEMENT OF THE PROBLEM AND METHOD OF SOLUTION

Suppose the magnetic field varies slowly with position (the change in a distance equal to the Larmor radius being small). For $x \rightarrow \pm\infty$ we have $H \rightarrow H_{\pm}$. The particle coming in from $-\infty$ has the magnetic moment I_-. As $x \rightarrow +\infty$, the magnetic moment of the particle will approach some other value I_+. We shall calculate the change $I_+ - I_-$. The result can be obtained qualitatively by the following simple method.

As is well known (cf. e. g., reference 4), if the curvature of the lines of force is neglected the problem reduces to the study of the model Hamiltonian

$$H = (p_x^2 + p_y^2 + m^2\omega^2(x)\, y^2)/2m, \tag{1}$$

where $\omega(x)$ is the Larmor frequency. The small parameter of the problem is the quantity $\alpha = r_L\omega'/\omega$, where r_L is the Larmor radius. Suppose the solution $x = x(t)$ is known for the coordinate x. Substituting this function into the equation of motion for y, we arrive at the problem of the change of the adiabatic invariant of an oscillator with frequency depending on the time, which was solved in reference 3. The result was found to be

$$\frac{\Delta I}{I} = 2\operatorname{Re}\left\{-i \exp 2i\left[\int^{t_0} \omega(t)\, dt + \varphi_-\right]\right\}, \tag{2}$$

where t_0 is a zero of the function $\omega(t)$ in the complex plane of t, and φ_- is the phase of the oscillator at $t \rightarrow -\infty$.* Since the adiabatic invariant changes only slightly during the motion, we shall treat it as a constant in obtaining the time dependence $x(t)$. Setting $I = \text{const}$ in the Hamiltonian (1) we get

$$dt = m dx / \sqrt{2m(E - I\omega(x))}. \tag{3}$$

Finally, substituting Eq. (3) in Eq. (2), we get

$$\frac{\Delta I}{I} = 2\operatorname{Re}\left\{-i \exp 2i\left[\int^{x_0} \frac{m\omega\, dx}{\sqrt{2m(E - I\omega)}} + \varphi_-\right]\right\}. \tag{4}$$

Here x_0 is the zero of the function $\omega(x)$.

Actually during the motion the adiabatic invariant is defined to within an amount α, and the formula (4) is true only apart from a factor of the order of unity multiplying the exponential function. To calculate this factor one would have to develop a perturbation theory. In view of the fact that the classical perturbation theory is extremely complicated, it is convenient to solve the quantum-mechanical problem first and then go over to the classical limit.

The Schrödinger equation for the Hamiltonian (1) is

$$\frac{1}{2m}(\Delta\psi - m^2\omega^2(x)\, y^2\psi) = -E\psi. \tag{5}$$

Here we have set $\hbar = 1$. To solve the problem, following reference 5, we introduce an orthogonal coordinate system in which the variables in Eq. (5)

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*Everywhere in the text expressions of the type $\int^x f(u)du$ are to be understood as meaning $\int_{-\infty}^x [f(u) - f(-\infty)]du + xf(-\infty)$.

"almost separate"

$$\xi = x - y^2\omega'/4\omega, \quad \eta = \sqrt{m\omega}\, y. \tag{6}$$

In these coordinates Eq. (5) takes the following form:

$$(L_0 + 2mE - m\omega\eta^2)\psi = -L_1\psi, \tag{7}$$

where

$$L_0 = m\omega\frac{\partial^2}{\partial\eta^2} + \sqrt{\omega}\frac{\partial}{\partial\xi}\left(\frac{1}{\sqrt{\omega}}\frac{\partial}{\partial\xi}\right); \tag{8}$$

$$\begin{aligned} L_1 = {} & \eta^2\left\{(f')^2 + \frac{1}{2}f''f\right\}L_0 + \frac{1}{f^2}\left[(f')^2 - \frac{1}{2}f''f\right]\frac{\partial}{\partial\eta}\left(\eta^2\frac{\partial}{\partial\eta}\right) \\ & - \frac{\eta^2}{f}\frac{\partial}{\partial\xi}\left[f\left((f')^2 - \frac{1}{2}f''f\right)\frac{\partial}{\partial\xi}\right] + O(x^4), \qquad f = (m\omega)^{-1/2}. \end{aligned} \tag{9}$$

2. THE ZEROTH APPROXIMATION

The zeroth-approximation is

$$\left[m\omega\left(\frac{\partial^2}{\partial\eta^2} - \eta^2\right) + \sqrt{\omega}\frac{\partial}{\partial\xi}\left(\frac{1}{\sqrt{\omega}}\frac{\partial}{\partial\xi}\right) + 2mE\right]\psi = 0. \tag{10}$$

Separating the variables in this equation, we get the set of solutions:

$$|n, E\rangle = Y_n(\eta)\, Z_{nE}(\xi),$$

where $Y_n(\eta)$ are the normalized eigenfunctions of an oscillator with frequency 1. It must be noted that there are three different cases, in which different expressions hold for the function $Z_{nE}(\xi)$:

1) The motion of the particle is infinite in both directions. In this case $Z_{nE}(\xi)$ has the form

$$\begin{aligned} Z_{nE}(\xi) = {} & \left(\frac{m}{2\pi k_{nE}}\right)^{1/2}(m\omega)^{1/4}\left\{\exp\left(i\int^{\xi} k_{nE}\, d\xi\right)\right. \\ & \left. + R_{nE}\exp\left(-i\int^{\xi} k_{nE}\, d\xi\right)\right\}, \end{aligned} \tag{11}$$

$$k_{nE}^2 = 2mE - m\omega(n + {}^1/_2).$$

2) There is one point on the real axis where $k_{nE} = 0$. This case corresponds to reflection from a "magnetic plug."

3) There are two real roots of the equation $k_{nE} = 0$. This corresponds to a particle that is in a "magnetic trap."

Only the first two cases are considered in the present paper. In the second case the function $Z_{nE}(\xi)$ has different forms on different sides of the turning point, and near the turning point it is a solution of the Airy equation. In this case, however, as Landau and Lifshitz have shown (cf. reference 6), one can use contour integration to calculate the matrix elements, and this leads to the same results as in the first case, which we shall treat here. The coefficient $k_{nE}(\xi)$ is of order of magnitude α on the interval where the important variation of ω occurs, and becomes exponentially small for $x \to +\infty$. The transverse quantum number n plays the role of the adiabatic invariant in the classical limit. In zeroth approximation there is no change of n.

3. THE SCATTERING MATRIX

Let us find the transition amplitude by perturbation theory

As is shown in reference 7, in calculating the matrix element $\langle n'E' | L_1 | nE \rangle$ we can keep only the first term in the expression for Z_{nE}. In the calculation of the matrix element one encounters integrals of the following type:

$$Q_{nE}^{n'E'} = \int_{-\infty}^{+\infty} \frac{\exp\{i(\rho + i\sigma)\}}{(k_{nE}k_{n'E'})^{1/2}}\frac{\omega'^2}{\omega^3}\, d\xi, \tag{12}$$

$$\rho + i\sigma \equiv \int^{\xi}(k_{nE} - k_{n'E'})\, d\xi. \tag{13}$$

The path of integration can be shifted into the half-plane in which $\sigma > 0$. In doing so one must carry the path around the singularities of the integrand. Then the branching associated with the vanishing of the quantity k_{nE} does not contribute to the required matrix element. In fact, near this point Eq. (12) loses its meaning; it must be replaced by a solution of Airy's equation which has no singularity at the point in question. The singularities of the integrand are the zeroes and poles of the function ω. The main contribution to the integral (12) will come either from the saddle point or else from the zero or pole ξ_0 of the function $\omega(\xi)$ that corresponds to the smallest value of $\sigma(\xi_0)$. For definiteness let us examine the case in which the contribution is either from the saddle point or from a simple zero.

At the saddle point ξ_1 we have $\omega(\xi_1) = (E - E')/(n - n')$. The points ξ_0 and ξ_1 lie on the curve $L\,[\mathrm{Im}\,\omega(\xi) = 0]$, which intersects the real axis at some point ξ'. If the quantity $(E - E')/(n - n')$ is positive, then ξ_1 lies on the curve L between the points ξ' and ξ_0. If, on the other hand, $(E - E')/(n - n') < 0$, then ξ_1 lies on the curve L beyond ξ_0. Let ω_m be the maximum value of $\omega(\xi)$ on the segment $(\xi'\xi_0)$ of the curve L. It is obvious that the condition for the existence of a saddle point is $(E - E')/(n - n') \le \omega_m$. Thus in the case $0 < (E - E')/(n - n') \le \omega_m$ the calculation of the integral (12) can be carried out by the ordinary method of steepest descents, which gives

$$Q_{nE}^{n'E'} = \left[\frac{\pi i}{(n-n')\,2mk_{nE}}\right]^{1/2} \frac{(\omega')^{3/2}}{\omega^3} \exp\{i\rho(\xi_1) - |\sigma(\xi_1)|\}. \tag{14}$$

The expression (14) cannot be applied in the cases E = E′ and n = n′. Indeed, in these cases the main contribution to the integral (12) comes from the residue at the zero ξ_0 of the function $\omega(\xi)$, which is a third-order pole of the integrand. In the case E = E′ the pole coincides with the saddle point. In the case n = n′ the pole does not coincide with the saddle point, but gives a larger contribution to the expression (12). Calculating the residue at ξ_0, we get

$$Q_{nE}^{n'E} = \pi \frac{n-n'}{4E} \exp\{i\rho(\xi_0) - |\sigma(\xi_0)|\}, \tag{15}$$

$$Q_{nE}^{nE'} = -\frac{\pi i}{A} \frac{(2m)^{1/2}}{(EE')^{1/4}} (\sqrt{E} - \sqrt{E'})^2 \exp\{i\rho(\xi_0) - |\sigma(\xi_0)|\}, \tag{16}$$

$$A = \omega'(\xi_0).$$

We recall that $\rho(\xi_0)$ and $\sigma(\xi_0)$ are taken from the definition (13) with the suitable values of nE, n′E′.

For

$$(E - E')/E \sim A^{1/2}(mE^3)^{-1/4} \sim \sqrt{\alpha}$$

the expression (14) goes over into the form (15). In the case $(E - E')/(n - n') < 0$ the integral (12) is given by the residue at the point ξ_0, which gives a result that coincides with Eq. (16). Using Eqs. (14), (15), (16) and the well known expressions for the matrix elements of an oscillator, we get

$$\langle n+2.\, E\,|\,L_1\,|\,n,\, E\rangle$$

$$= \frac{3\pi}{16}\sqrt{(n+1)(n+2)} \quad \exp\{i\rho(\xi_0) - |\sigma(\xi_0)|\},$$

$$\langle n-2.\, E\,|\,L_1\,|\,n,\, E\,\rangle$$

$$= \frac{3\pi}{16}\sqrt{(n-1)n} \exp\{-i\rho(\xi_0^*) - |\sigma(\xi_0)|\}. \tag{17}$$

For transitions with changes of n by more than 2 the results are exponentially small in comparison with those just stated.

For $(E - E')/(n - n') \gg \alpha^{1/2}$ the matrix element has the structure (14) with a somewhat different coefficient of the exponential function, which coefficient, however, is less than or of the order of unity everywhere in the domain of applicability.

It is shown in the Appendix that the subsequent approximations make contributions to the scattering matrix that are small in α. Thus the nondiagonal transition matrix elements reduce to the expressions (17). In virtue of the unitarity relation the diagonal element of the scattering matrix is of the form $a_{nn} = e^{i\varphi}[1 + 0(\exp\{-2|\sigma(\xi_0)|\}]$, where φ is a small quantity, $\varphi \lesssim \alpha$.

Knowing the scattering matrix, one can easily calculate the change of the adiabatic invariant of the particle. For this it is necessary to construct a wave packet describing the classical particle on its trajectory. Carrying out calculations analogous to those done in reference 3, we get

$$\frac{\Delta I}{I} = \left(\frac{3\pi}{8}\right) 2\,\mathrm{Re}\left[-i \exp 2i\left\{\int^{\xi_0} \frac{m\omega\, d\xi}{\sqrt{2m(E - I\omega)}} + \varphi_-\right\}\right]. \tag{18}$$

We note that the coefficient of the exponential function does not depend on I and E. Within the framework of our method it is not hard to take into account the curvature of the lines of force. This problem, however, needs further investigation.

APPENDIX

Let us estimate the contribution to elements of the scattering matrix made by the second approximation. Having in mind the application to the classical case, we confine ourselves to just the calculation of the matrix element $a_{n,\,n+2}$ (cf. reference 6). The second approximation of perturbation theory is given by the formula

$$\int \frac{\langle n, E\,|\,L_1\,|\,n+2, E'\rangle \langle n+2, E'\,|\,L_1\,|\,n+2, E\rangle + \langle n, E\,|\,L_1\,|\,n, E'\rangle \langle E', n\,|\,L_1\,|\,n+2, E\rangle}{E' - E - i\delta}\, dE'. \tag{A. 1.}$$

Let us consider for definiteness the first term. We divide the integration over E′ into three intervals: (0, E), $(E, E + 2\omega_m)$ and $(E + 2\omega_m, \infty)$. In the interval (0, E) the integrand is of the form [cf. Eq. (16)]

$$\frac{(\sqrt{E} - \sqrt{E'})^4}{A^2\sqrt{EE'}\,(E' - E)} \exp\left\{i\int^{\xi_0} (k_{nE} + k_{n+2,\,E} - 2k_{n+2,\,E'})\, d\xi\right\}. \tag{A. 2}$$

The integrand has a triple zero at the point E′ = E, and therefore the problem of passage around a singularity does not arise in the integration. At this same point the exponent has a maximum value which is equal to the exponent obtained in the first approximation. Therefore the main contribution comes from a region of the order α near E′ = E. Integration over this region gives a coefficient for the exponential of the order α. Actually the expression (A. 2) ceases to be valid in a neighborhood of E′ = E of the order $\alpha^{1/2}$. For the estimate in this region we replace the matrix element

$\langle n, E \mid L_1 \mid n+2, E \rangle$ by the quantity (17). In this region the matrix element $\langle n+2, E' \mid L_1 \mid n+2, E \rangle$ is at least of the order α. It is not hard to see that in this case the integration in Eq. (A. 1) gives for the exponential function a coefficient of the order of or smaller than $\alpha^{1/2}$.

In this same way it can be shown that the integration over the interval $(E, +\infty)$ leads to this same estimate.

It is not hard to carry out analogous arguments for the subsequent approximations of perturbation theory.

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THE DECAY $\Sigma^0 \rightarrow \Lambda^0 + 2\gamma$ AND THE MAGNETIC MOMENT OF THE Σ^0 HYPERON

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The possibility of determing the magnetic moment of the Σ^0 hyperon by comparing the decays $\Sigma^0 \rightarrow \Lambda^0 + 2\gamma$ and $\Sigma^0 \rightarrow \Lambda^0 + \gamma$ is examined.

BESIDES the known decay

$$\Sigma^0 \rightarrow \Lambda^0 + \gamma \qquad (1)$$

there must also occur the decay

$$\Sigma^0 \rightarrow \Lambda^0 + 2\gamma. \qquad (2)$$

The present paper is devoted to an estimate of the probability of the decay (2). As will be shown below, the ratio of the probabilities of the decays (2) and (1) is very small ($\sim 10^{-6}$) and depends strongly on the values of the magnetic moments of the Λ and Σ hyperons. The value of the magnetic moment of the Λ^0 hyperon will apparently be determined in the near future by measuring the spin flip of the Λ^0 hyperon in a strong magnetic field. This method cannot be applied in the case of the Σ^0 hyperon, however, because of the short lifetime of the latter. Using isotopic invariance, one can find the magnetic moment of the Σ^0 hyperon from the known relation $2\mu_{\Sigma_0} = \mu_{\Sigma^+} + \mu_{\Sigma^-}$, which was first obtained by Marshak, Okubo, and Sudarshan,[1] if the magnetic moments of the Σ^- and Σ^+ hyperons are known. Below we shall discuss the possibilities of a direct measurement of the magnetic moment of the Σ^0 hyperon using the decay (2).*

The process (1) is described by the Feynman graph of Fig. 1; the circle in this graph represents the totality of the virtual strong interactions. The matrix element corresponding to this graph has the form

$$M_1 = \mu \bar{u}_2 \hat{\epsilon} \hat{k} O u_1, \qquad (3)$$

where the coefficient μ may be called the magnetic moment of the transition $\Sigma \rightarrow \Lambda$; u_1 and u_2 are the spinors of the Σ and Λ hyperon, respectively; ϵ and k are the four-vectors of the polarization and the momentum of the photon ($\hat{k} = k_0\gamma_0 - \mathbf{k}\boldsymbol{\gamma}$, where the γ are the Feynman matrices). The form of the operator O depends on

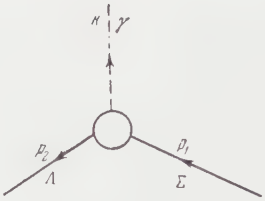

FIG. 1

the relative parity of the Σ and Λ hyperons, which we shall denote by $P_{\Sigma\Lambda}$. If $P_{\Sigma\Lambda} = -1$, we have $O = \gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3$; if $P_{\Sigma\Lambda} = +1$, we have $O = 1$. The probability of the decay (1), calculated with the help of the matrix element M_1, is independent of $P_{\Sigma\Lambda}$ and is equal to*

$$w_1 = 4\mu^2\Delta^3, \qquad (4)$$

where $\Delta = m_{\Sigma^0} - m_{\Lambda^0} \approx 76$ Mev (here and in the following we neglect terms of order Δ/m_Σ compared to unity). Assuming that $\mu = e/2m_\pi$, where m_π is the mass of the π meson, we obtain from formula (4) $\tau = 1/w_1 \approx 4 \times 10^{-21}$ sec; if we choose $\mu = e/m_p$, then $\tau \approx 4.5 \times 10^{-20}$ sec.

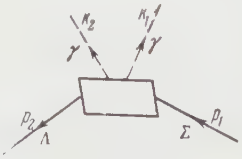

FIG. 2

Let us now turn to the decay (2). It is described, in its most general form, by the graph of Fig. 2. The box in this graph represents symbolically the set of all graphs which give a contribution to the decay (2). In the following we shall consider only a few of these graphs (Fig. 3, a and b) which are believed to give the most important contribution.

The emission of one of the photons in these graphs is, as before, due to the interaction of Fig. 1 (transition $\Sigma \rightarrow \Lambda$); the emission of the other photon is connected with the magnetic mo-

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*In this connection it is also of interest to investigate the internal bremsstrahlung which accompanies the production of Σ^0 hyperons.

*We use a system of units in which $\hbar = c = 1$ and $e^2 = 1/137$. In the usual units $1/m_\pi = 4.7 \times 10^{-24}$ sec.

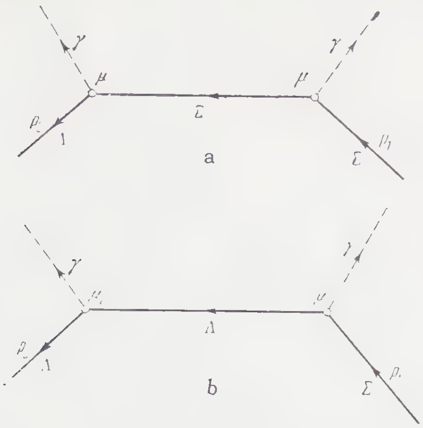

FIG. 3

ment of the Σ^0 hyperon μ_1 (Fig. 3a) and the magnetic moment of the Λ^0 hyperon μ_2 (Fig. 3b).

The graphs of Figs. 3a and 3b differ from the remaining graphs symbolically shown in Fig. 2 in that the energy dependence of the emitted photons is different. This is connected with the presence of pole terms* in the graphs of Fig. 3 which give contributions which are inversely proportional to the energy of the photon ($\sim 1/\Delta$). The remaining graphs contain, besides the term $\sim 1/\Delta$, a term of order $\sim R_0$, where R_0 is a measure of the size of the radiating region ($1/m_\pi \gtrsim R_0 \gtrsim 1/m_p$). If the mass difference of the Σ and Λ hyperons were so small in comparison with the mass of the π meson that the inequality $R_0\Delta \ll 1$ were fulfilled, the graphs 3a and 3b would, then, definitely give the most important contribution. In actuality, $\Delta/m_\pi \approx 1/2$, and the question whether the graphs 3a and 3b are the basic ones must really be decided by putting in specific numbers. This conclusion is confirmed by the computation of the contribution of the graph of Fig. 4 with a radiating region of an effective size of the order $\sim 1/m_\pi$ (see Appendix II).

The matrix element corresponding to the sum of the graphs 3a and 3b, symmetrized with respect to the photons, is equal to

$$M_2 = \mu\mu_1\bar{u}_2 O\left[\hat{e}_2\hat{k}_2\frac{1}{\hat{p}_1-\hat{k}_1-m_1}\hat{e}_1\hat{k}_1+\hat{e}_1\hat{k}_1\frac{1}{\hat{p}_1-\hat{k}_2-m_1}\hat{e}_2\hat{k}_2\right]u_1$$
$$+\mu\mu_2\bar{u}_2\left[\hat{e}_2\hat{k}_2\frac{1}{\hat{p}_2+\hat{k}_2-m_2}\hat{e}_1\hat{k}_1+\hat{e}_1\hat{k}_1\frac{1}{\hat{p}_2+\hat{k}_1-m_2}\hat{e}_2\hat{k}_2\right]Ou_1. \quad (5)$$

Here k_1, k_2 and e_1, e_2 are the momenta and the polarization vectors of the photons.

The total probability for the decay (2), calculated with the help of M_2, is equal to (see Appendix I)

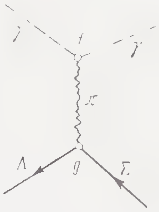

FIG. 4

$$w_2^- = (2\Delta^5/5\pi)\mu^2(\mu_1^2+\mu_2^2+\tfrac{2}{3}\mu_1\mu_2), \quad (6)$$

for $P_{\Sigma\Lambda} = -1$, and equal to

$$w_2^+ = (14\Delta^5/45\pi)\mu^2(\mu_1^2+\mu_2^2-\tfrac{6}{7}\mu_1\mu_2), \quad (7)$$

for $P_{\Sigma\Lambda} = +1$.

If we include only the contributions from the graphs 3a and 3b, we obtain for the ratio of the probabilities of the decays (2) and (1)

$$w_2^-/w_1 = (\Delta^2/10\pi)(\mu_1^2+\mu_2^2+\tfrac{2}{3}\mu_1\mu_2) \text{ for } P_{\Sigma\Lambda} = -1; \quad (8)$$

$$w_2^+/w_1 = (7\Delta^2/90\pi)(\mu_1^2+\mu_2^2-\tfrac{6}{7}\mu_1\mu_2) \text{ for } P_{\Sigma\Lambda} = +1. \quad (9)$$

We see that this ratio does not depend on μ at all, and depends weakly on the relative parity of the Σ and Λ hyperons.

It follows from (8) and (9) that the value of μ_{Σ^0} can be determined from the ratio w_2/w_1, if the relative parity $P_{\Sigma\Lambda}$ and the value of the magnetic moment of the Λ^0 hyperon, μ_2, are known (two solutions are possible here).

Let us consider a crude model in which the Σ^0 hyperon is represented by a Λ^0 hyperon with a π^0 meson rotating about it. In this model $\mu_1 = \mu_2$ for $P_{\Sigma\Lambda} = -1$ and $\mu_1 = -\mu_2$ for $P_{\Sigma\Lambda} = +1$. Substituting this in (8) and (9), we obtain

$$w_2^-/w_1 = 4\Delta^2\mu_1^2/15\pi, \quad (10)$$

$$w_2^+/w_1 = 2\Delta^2\mu_1^2/9\pi. \quad (11)$$

If we now assume that $\mu_1 = e/m_p$, we find

$$w_2/w_1 \approx 5\cdot 10^{-6}. \quad (12)$$

As we shall see presently, even this small ratio is apparently an overestimate.

For this purpose let us estimate the expected value of the magnetic moments of the Λ^0 and Σ^0 hyperons. Both particles have vanishing normal magnetic moments. We are therefore concerned only with their anomalous magnetic moments due to charged virtual particles.

It is well known that the anomalous magnetic moments of baryons are in general equal to the sum of an isoscalar and an isovector term:

$$\mu = \mu_S + T_3\mu_V. \quad (13)$$

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*For a detailed discussion of the role of the pole terms in the emission of quanta of small energy, cf. the paper of Low.[2]

Thus we have for nucleons

$$\mu_p = \mu_S + \mu_V = 1{,}79 \text{ nucl. magnetons},$$

$$\mu_n = \mu_S - \mu_V = 1{,}91 \text{ nucl. magnetons},$$

which leads to $\mu_S = -0.06$ nucl. magnetons and $\mu_V = 1.85$ nucl. magnetons; therefore $\mu_S/\mu_V \sim 3\%$. It can be easily shown that only the isoscalar part is different from zero for Λ^0 and Σ^0 hyperons. In the case of the Σ^0 hyperon this is seen by setting $T_3 = 0$ in (13); in the case of the Λ^0 hyperon it is an obvious consequence of the fact that the isospin of the Λ hyperon is zero. If we assume that $\mu_S \ll \mu_V$ for the hyperons as well as for the nucleons, we should expect that the magnetic moments of the Λ^0 and Σ^0 hyperons are about an order of magnitude smaller than the anomalous magnetic moments of the Σ^+ and Σ^- hyperons, and the latter should, according to formula (13), have about equal magnitude and opposite sign.

The inequality $\mu_S \ll \mu_V$ follows readily from the assumption that the main contribution to the anomalous magnetic moments of the nucleons is due to the virtual π mesons. This assumption is quite realistic, since the π mesons are the lightest among the strongly interacting particles, and it is these which determine the size of the cloud of virtual particles surrounding the nucleon. Since the interaction of the π mesons with the electromagnetic field is of the isotopic vector type, we obtain $\mu_S = 0$, $\mu_V \neq 0$ in the approximation in which the electric charges of all virtual particles except the π mesons are "excluded." In order of magnitude we should have $\mu_V \lesssim e/2m_\pi$. The approximation in which only the electric charges of the π mesons (and of the Σ hyperons, whose interaction with the electromagnetic field is also of the isotopic vector type) are "included" will be called the isovector approximation.

This approximation was first considered by Katsumori;[3] he formulated his results in the form of a "mirror theorem," according to which

$$\mu_p + \mu_n = 0, \quad \mu_{\Xi^-} + \mu_{\Xi^0} = 0,$$

$$\mu_{\Sigma^-} + \mu_{\Sigma^+} = 2\mu_{\Sigma^0} = 0, \quad \mu_\Lambda = 0.$$

It is easily seen that in the isovector approximation μ, the magnetic moment of the transition $\Sigma \to \Lambda + \gamma$, is different from zero, unlike μ_{Λ^0} and μ_{Σ^0}. In the isovector approximation the decay $\Sigma^0 \to \Lambda^0 + \gamma$ is therefore allowed, while the decay $\Sigma^0 \to \Lambda^0 + 2\gamma$ is forbidden (since $\mu_\Lambda = \mu_{\Sigma^0} = 0$). It may turn out that this forbiddenness comes only from the special form of the graphs 3a and 3b and that there exist other graphs which give a non-vanishing contribution to the matrix element of the decay (2) in the isovector approximation. If this were so, then our starting assumption that the graphs 3a and 3b are the most important ones would be incorrect. It can be easily shown, however, that not only the graphs 3a and 3b, but also the most general graph 2, gives a vanishing contribution in the isovector approximation. Indeed, the isotopic structure of the matrix element corresponding to the graph 2 has in the isovector approximation the form

$$\langle \bar{0} | 1 | n \rangle \langle \bar{n} | 1 | 1 \rangle, \tag{14}$$

where $\langle \bar{f} | 1 | i \rangle$ is the matrix element for the transition from the state i to the state f due to an interaction with $T = 1$; $|1\rangle$ and $\langle \bar{0}|$ designate the isospin of the initial (Σ) and final (Λ) states; n is the isospin of the intermediate states. The matrix element $\langle \bar{0} | 1 | n \rangle$ is different from zero only if $n = 1$; but for $n = 1$ the matrix element $\langle \bar{n} | 1 | 1 \rangle$ vanishes, since $T_3 = 0$ for the initial and final states. The decay $\Sigma^0 \to \Lambda^0 + 2\gamma$ is therefore strictly forbidden in the isovector approximation.

Let us now turn to the estimate of the ratio w_2/w_1. On the basis of the above-mentioned isospin considerations we may conclude that this ratio should be of order 10^{-7}. Such a small value for the ratio w_2/w_1 makes it unlikely that the decay $\Sigma^0 \to \Lambda^0 + 2\gamma$ can be observed with present experimental techniques.

The authors take this opportunity to express their gratitude to V. B. Berestetskiĭ, S. M. Bilen'kiĭ, B. L. Ioffe, and I. Ya. Pomeranchuk for valuable comments.

APPENDIX I

Let us compute the spectra of the secondary particles in the decay (2). Summing over the spins of the photons and the Λ^0 hyperon and averaging over the spin of the Σ^0 hyperon in the standard way, we obtain for the differential probability of the decay (2)

$$w_2 = \frac{2}{(2\pi)^3 m_1} \overline{|M_2|^2} \delta^4 (p_1 - p_2 - k_1 - k_2) \frac{d\mathbf{k}_1}{\omega_1} \frac{d\mathbf{k}_2}{\omega_2} \frac{d\mathbf{p}_2}{E_2}, \tag{I.1}$$

$$\begin{aligned} \overline{|M_2|^2} = {} & \mu^2\mu_1^2 \Big[(p_1k_1)(p_2k_1) + (p_1k_2)(p_2k_2) - m_1(m_1 \pm m_2)(k_1k_2) \\ & + \frac{m_1^3(m_1 \pm m_2)(k_1k_2)^2}{2(p_1k_1)(p_1k_2)}\Big] + \mu^2\mu_2^2 \Big[(p_1k_1)(p_2k_1) + (p_1k_2)(p_2k_2) \\ & - m_2(m_2 \pm m_1)(k_1k_2) + \frac{m_2^3(m_2 \pm m_1)(k_1k_2)^2}{2(p_2k_1)(p_2k_2)}\Big] \\ & + \mu^2\mu_1\mu_2 \Big\{ \mp [2(p_1k_1)(p_1k_2) + 2(p_2k_1)(p_2k_2) \\ & - (k_1k_2)(m_1 \mp m_2)^2] + m_1m_2(k_1k_2) \Big[(p_1k_1)(p_1k_2) \\ & + (p_2k_1)(p_2k_2) - \frac{1}{2}(k_1k_2)(m_1^2 + m_2^2)\Big] \\ & \times \Big[\frac{1}{(p_1k_1)(p_2k_1)} + \frac{1}{(p_1k_2)(p_2k_2)}\Big]\Big\}, \end{aligned} \tag{I.2}$$

where the upper signs correspond to the case $P_{\Sigma\Lambda} = +1$ and the lower signs to the case $P_{\Sigma\Lambda} = -1$.

The further calculations will be carried out with an accuracy up to order Δ/m. The photon spectrum has the form

$$w(\omega)\, d\omega = 4\mu^2\pi^{-1}\left[\mu_1^2 + \mu_2^2 + \tfrac{2}{3}\mu_1\mu_2\right] \times \left[\omega^2 + (\Delta - \omega)^2\right]\omega(\Delta - \omega)\, d\omega, \tag{I.3}$$

for $P_{\Sigma\Lambda} = -1$, and

$$w(\omega)\, d\omega = 4\mu^2\pi^{-1}\{[(\mu_1^2 + \mu_2^2)(\omega^2 + (\Delta - \omega)^2) - \tfrac{2}{3}\omega(\Delta - \omega)] + \mu_1\mu_2[\tfrac{2}{3}(\omega^2 + (\Delta - \omega)^2) - 4\omega(\Delta - \omega)]\}\omega(\Delta - \omega)\, d\omega, \tag{I.4}$$

for $P_{\Sigma\Lambda} = +1$.

The angular distribution of the photons is given by

$$w(\cos\vartheta)\, d\cos\vartheta = (\mu^2\Delta^5/5\pi)\left[\mu_1^2 + \mu_2^2 + \mu_1\mu_2\left(1 + \tfrac{4}{3}\cos\vartheta - \cos^2\vartheta\right)\right] d\cos\vartheta, \tag{I.5}$$

if $P_{\Sigma\Lambda} = -1$, and by

$$w(\cos\vartheta)\, d\cos\vartheta = (\mu^2\Delta^5/15\pi)\left[(\mu_1 + \mu_2)^2(2 + \cos^2\vartheta) - \mu_1\mu_2(1 + 3\cos^2\vartheta)\right] d\cos\vartheta, \tag{I.6}$$

if $P_{\Sigma\Lambda} = +1$.

The spectrum of the Λ hyperons has the form

$$w^-(x)\, dx = \frac{\mu^2\Delta^5}{\pi}\left[(\mu_1^2 + \mu_2^2)\left(1 + \frac{x^2}{3}\right) + 2\mu_1\mu_2\left(-5 + \frac{17}{3}x^2 + \frac{2(1 - x^2)}{x}\ln\frac{1 + x}{1 - x}\right)\right] x^2 dx, \tag{I.7}$$

for $P_{\Sigma\Lambda} = -1$, and

$$w^+(x)\, dx = \frac{\mu^2\Delta^5}{\pi}\left[(\mu_1^2 + \mu_2^2)\left(-1 + \frac{7}{3}x^2 + \frac{(1 - x^2)^2}{x}\ln\frac{1 + x}{1 - x}\right) + 2\mu_1\mu_2\left(-5 + \frac{13}{3}x^2 + \frac{2(1 - x^2)}{x}\ln\frac{1 + x}{1 - x}\right)\right] x^2 dx, \tag{I.8}$$

for $P_{\Sigma\Lambda} = +1$. Here $x = p/\Delta$ (p is the momentum of the Λ hyperon).

Integrating formulas (I.3) to (I.8) we easily obtain the formulas (6) and (7) quoted in the text.

APPENDIX II

Let us now estimate the contribution of the graph 4 to the probability of the decay $\Sigma \rightarrow \Lambda + 2\gamma$.

This graph describes the process (2) as going through the decay of a virtual π^0 meson. If $P_{\Sigma\Lambda} = +1$, the virtual π^0 meson is in a P state and the contribution of the graph 4 is negligibly small. Therefore, we consider only the case $P_{\Sigma\Lambda} = -1$. The matrix element corresponding to the graph 4 is

$$M_2' = 4\pi i \frac{fg}{m_\pi}\bar{u}_2 u_1\left[(k_1 + k_2)^2 - m_\pi^2\right]^{-1}\varepsilon_{\alpha\beta\gamma\delta}\, e_{1\alpha}e_{2\beta}k_{1\gamma}k_{2\delta}, \tag{II.1}$$

where m_π is the mass of the π meson, g is the constant of the strong $\Sigma\Lambda\pi$ coupling, and the dimensionless constant f characterizes the decay of the π^0 meson ($\pi^2 f^2 m_\pi \tau_0 = 1$, where τ_0 is the lifetime of the π^0 meson). If both M_2' and M_2 [formula (5)] are included, we obtain for w_2^-, instead of formula (6),

$$w_2^- = \frac{2\Delta^5}{5\pi}\mu^2\left(\mu_1^2 + \mu_2^2 + \frac{2}{3}\mu_1\mu_2\right) + \frac{32\pi}{105}f^2g^2\frac{\Delta^7}{m_\pi^6}C_1 - \frac{8}{90}fg\,\mu(\mu_1 - \mu_2)\frac{\Delta^6}{m_\pi^3}C_2, \tag{II.2}$$

The coefficients C_1 and C_2 are corrections which take account of the term $(k_1 + k_2)^2$ together with m_π^2 in the π-meson Green's function:

$$C_1 = \frac{105}{16\lambda^7}\left[\frac{2}{3}\lambda^3 - 5\lambda + \frac{5 - 4\lambda^2}{(1 - \lambda^2)^{1/2}}\tan^{-1}\frac{\lambda}{(1 - \lambda^2)^{1/2}}\right] = 1{,}6. \tag{II.3}$$

$$C_2 = \left[\frac{45}{8\lambda^5}\int_0^\lambda \frac{\ln[1 - \lambda^2 + z^2]}{z^2 - \lambda^2}dz - \frac{15}{8\lambda^2} - \frac{45}{8\lambda^4}\right] = 1.2, \tag{II.4}$$

where $\lambda = \Delta/m_\pi$.

Choosing $\tau_0 = 10^{-16}$ sec, $\mu \sim e/m_\pi$, and $\mu_\Lambda \sim \mu_{\Sigma^0} \sim e/10m_p$, we find that the contribution of the graph 4 becomes important when $g \approx 1$.

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[1] Marshak, Okubo, and Sudarshan, Phys. Rev. **106**, 599 (1957).

[2] F. E. Low, Phys. Rev. **110**, 974 (1958).

[3] H. Katsumori, Progr. Theor. Phys. **18**, 375 (1957).

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ON THE SUPERCONDUCTIVITY OF A FERROMAGNETIC WITH A WEAK EXCHANGE INTERACTION

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We discuss the influence of s-d exchange on the condition for the existence of a superconducting state in a ferromagnet, and also on its characteristics such as the critical temperature, critical field, and the specific heat.

IT was shown in a paper by the present authors[1] that the establishment of superconductivity in ferromagnetics of the type Fe, Co, and Ni is prevented by the shift in the Fermi spheres of the s-conduction electrons with opposite spins, a shift caused by their coupling with the d or f electrons of the interior shells which have no spin-saturation, i.e., caused by the (s-d)-exchange interaction. It was also shown that superconductivity can occur in ferromagnetics with a sufficiently weak (s-d)-exchange coupling ($\mu J \ll \hbar\omega$, where μ is the excess of d or f electrons with the predominant spin orientation per lattice site;* J is the energy parameter of the (s-d) exchange, which is approximately independent of the wave vector **k** (reference 2); ω is the average phonon frequency). The existence of a superconducting ferromagnet was apparently established by Matthias, Suhl, and Corenzwit[3] in the (Ce, Gd)Ru_2 system. In this connection it is of interest to consider in more detail the influence of the above-mentioned shift on the characteristics of the superconducting state, and this is the aim of the present paper.†

1. THE GROUND STATE AND FREE ENERGY OF A FERROMAGNETIC SUPERCONDUCTOR

When a ferromagnetic superconductor has an excess of s electrons with spins oriented to the left due to the magnetizing effect of the d or the f electrons, one can write the Hamiltonian of the conduction electrons, between which there are interactions induced by the phonons[4] and the ferromagnons[6] and also a Coulomb repulsion, in the form[4,2]

$$\mathscr{H} = \sum_{k>\overset{+}{k}_F} \overset{+}{\varepsilon}_k \overset{+}{n}_k + \sum_{k>\overset{-}{k}_F} \overset{-}{\varepsilon}_k \overset{-}{n}_k + \sum_{k<\overset{+}{k}_F} |\overset{+}{\varepsilon}_k|(1-\overset{+}{n}_k) + \sum_{k<\overset{-}{k}_F} |\overset{-}{\varepsilon}_k|(1-\overset{-}{n}_k) - \sum_{kk'} V_{kk'}\overset{+}{b}_{k'}b_k, \qquad (1)$$

where $\epsilon_k^+ = E_k + \frac{1}{2}\mu J - E_F$ and $\epsilon_k^- = E_K - \frac{1}{2}\mu J - E_F$ are the energies, relative to the Fermi energy E_F, of electrons F with spins oriented to the right (+) and to the left (−), respectively; n_k^+ and n_k^-, k_F^+ and k_F^- are respectively the occupation numbers and maximum wave vectors at T = 0, when there is no interaction;

$$\overset{+}{b}_k = \overset{+}{c}{}^{+}_{k}\overset{+-}{c}_{-K}, \quad b_k = \overset{-}{c}_{-k}\overset{+}{c}_k$$

are the pair creation and annihilation operators, expressed in terms of the usual Fermi operators for electrons $\bar{c}$ and $\overset{+}{c}$; $V_{kk'}$ are the matrix elements of the interaction which will be assumed to be constant,[4] equal to V within a narrow region $|\epsilon_k| = |E_k - E_F| < \hbar\omega$, and equal to zero outside that region. In order that superconductivity exist, it is necessary[6] to have $V = V_p - V_c - V_{sd} = 0$ [V_p, V_c, and V_{sd} are respectively the matrix elements of the interelectronic interaction induced by the phonons, the quasi-Coulomb interaction, and the effective repulsion between conduction electrons induced by s-d exchange]. Since $V_{sd} \sim J^2$ (reference 7) and the influence of the shift in the Fermi sphere is a first-order effect in the parameter J (references 1 and 2), we assume in the case of weak (s-d) exchange that although the quantity V_{sd} decreases the constant V, it still leaves it positive.

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*μ is the relative magnetization of the d or f electron per lattice site. This quantity is weakly temperature dependent in the low temperature region.

†We use in the following the Bardeen-Cooper-Schrieffer scheme.[4] We could use also the more rigorous Bogolyubov scheme,[5] but it is then necessary to modify the canonical transformations in order to take the Fermi shift into account.

Since in our case, apart from pairs of Fermi particles (k +, − k −), there must already exist in the ground state "left-hand singlets" (i.e., electrons with − spin in the state − k without electrons with + spin in the state k), one must write the wave function in the form

$$\Psi = \prod_k [(1-h_k)^{1/2} + h_k^{1/2}\overset{+}{b}_k] \prod_{k' \neq k} \overset{+-}{c}_{k'}\Phi_0, \tag{2}$$

where Φ_0 is the "vacuum" of the system, and h_k the probability for the production of a pair (k+ , − k −).

According to (2) the expansions

$$\Psi = h_k^{1/2}\varphi_1 + (1-h\ )^{1/2}\varphi_0, \tag{3}$$

occur for any given k if k corresponds to a state of basic pairs;

$$\Psi = (h_k h_{k'})^{1/2}\varphi_{11} + [h_k(1-h_{k'})]^{1/2}\varphi_{10} + [(1-h_k)h_{k'}]^{1/2}\varphi_{01} + [(1-h_k)(1-h_{k'})]^{1/2}\varphi_{00}, \tag{4}$$

if k and k′ belong to states of basic pairs; and

$$\Psi = \overset{+-}{c}_k\varphi_0, \tag{5}$$

if k belongs to a state of "left-hand singlets." Here, φ_1 is a state with a pair (k +, − k −), φ_0 the state when there is no such pair, φ_{11} the state with pairs (k +, − k −; k′ +, − k′ −), φ_{00} a state when there are no such pairs, φ_{01} the state with a pair in the k′ state and no pair in the k state, and φ_{10} the state with a pair in k and without a pair in k′. If s_k^- is the probability that the state k is occupied by a left-hand singlet, the expansions (3) to (5) correspond respectively to probabilities $(1 - s_k^-)$, $(1 - s_k^-)(1 - s_{k'}^-)$, and s_k^-. By analogy with Bardeen, Cooper, and Schrieffer's paper[4] we get from the condition that the energy be a minimum an expression for h_k:

$$h_k = {}^1/_2[1 - \varepsilon_k/(\varepsilon_k^2 + \varepsilon_0^2)^{1/2}], \tag{6}$$

where unlike in reference 4

$$\varepsilon_0 = V\sum_k [h_k(1-h_k)]^{1/2}(1-s_k^-) \tag{7}$$

(so that only states with $s_k^- \neq 1$ give a contribution to ϵ_0).

The expression for the average energy of the system will be of the form

$$\begin{aligned} W_0 = &-\sum_{\mathrm{III}}\{(\varepsilon_k^2+\varepsilon_0^2)^{1/2} - \varepsilon_k - \varepsilon_0^2[2(\varepsilon_k^2+\varepsilon_0^2)]^{-1/2}\} \\ &+\sum_{\mathrm{III}} s_k^-\{(\varepsilon_k^2+\varepsilon_0^2)^{1/2} - \frac{\mu J}{2} - \varepsilon_0^2[2(\varepsilon_k^2+\varepsilon_0^2)]^{-1/2}\} \\ &-\sum_{\mathrm{II}}(1-s_k^-)\{(\varepsilon_k^2+\varepsilon_0^2)^{1/2} - \frac{\mu J}{2} - \varepsilon_0^2[2(\varepsilon_k^2+\varepsilon_0^2)]^{-1/2}\} \\ &-\sum_{\mathrm{I}}\{(\varepsilon_k^2+\varepsilon_0^2)^{1/2} + \varepsilon_k + \varepsilon_0^2[2(\varepsilon_k^2+\varepsilon_0^2)]^{-1/2}\} \\ &+\sum_{\mathrm{I}} s_k^-\{(\varepsilon_k^2+\varepsilon_0^2)^{1/2} - \frac{\mu J}{2} - \varepsilon_0^2[2(\varepsilon_k^2+\varepsilon_0^2)]^{-1/2}\}, \end{aligned} \tag{8}$$

where I, II, and III denote respectively the regions of summation with $k < k_F^+$, $k_F^+ < k < k_F^-$, and $k > k_F^-$. The region II has then clearly an energy width μJ and contains within it the Fermi surface E_F. Since

$$(\varepsilon_k^2+\varepsilon_0^2)^{1/2} - \varepsilon_k - \varepsilon_0^2[2(\varepsilon_k^2+\varepsilon_0^2)]^{-1/2} > 0$$

and even more strongly in I and III

$$(\varepsilon_k^2+\varepsilon_0^2)^{1/2} - \mu J/2 - \varepsilon_0^2[2(\varepsilon_k^2+\varepsilon_0^2)]^{-1/2} > 0$$

(because $|\epsilon_k| > {}^1/_2\mu J$ in I and III), according to (8) there will be no left-hand singlets in I and III (i.e., $s_k^- = 0$), which is of great advantage. The absence of left-hand singlets in II, however, and the corresponding presence of such singlets in I (or III) is energetically unprofitable because of the inequality

$$\{(\varepsilon_k^2+\varepsilon_0^2)^{1/2} - \frac{\mu J}{2} - \varepsilon_0^2[2(\varepsilon_k^2+\varepsilon_0^2)]^{-1/2}\}_{\mathrm{I}} > \{(\varepsilon_k^2+\varepsilon_0^2)^{1/2} - \frac{\mu J}{2} - \varepsilon_0^2[2(\varepsilon_k^2+\varepsilon_0^2)]^{-1/2}\}_{\mathrm{II}}, \tag{9}$$

where the expression with index I refers to any state from the region I, and the index II to the region II. There corresponds thus to the ground state a distribution of pairs in the regions I and III, according to (6), and an occupation of region II by the left-hand singlets ($s_{k,\mathrm{II}}^- = 1$), so that the summation in (7) extends only over regions I and III (with $s_k^- = 0$). It follows therefore from (7) and (6) that

$$[N(0)V]^{-1} = \int_{\mu J/2}^{\hbar\omega}(\varepsilon^2+\varepsilon_0^2)^{-1/2}d\varepsilon, \tag{10}$$

where N(0) is the density of states near the Fermi surface.

Using (10) we find instead of (7) and (8)

$$\varepsilon_0 = [2\sinh(1/N(0)V)]^{-1}\{[2\hbar\omega e^{-1/N(0)V} - \mu J] \times [2\hbar\omega e^{1/N(0)V} - \mu J]\}^{1/2}, \tag{11}$$

$$W_0 = -N(0)[2\hbar\omega - \mu J e^{1/N(0)V}]^2/2[e^{2/N(0)V} - 1]. \tag{12}$$

Equations (11) and (12) go over into Eqs. (2.40) and (2.42) of reference 4 when $\mu J = 0$. When μJ increases from 0 to $2\hbar\omega e^{-1/N(0)V}$ the quantity ϵ_0 decreases to zero and the energy of the ground state W_0 increases to $W_n = 0$, i.e., to the energy of the normal state. Superconductivity is thus possible only for weak (s-d) coupling, provided that

$$\mu J < 2\hbar\omega e^{-1/N(0)V}. \tag{13}$$

The quantity ϵ_0 determines the dispersion law for the excitations which can be formed in the case under consideration, firstly by the breaking up in I or III of a pair (k_1 +, − k_1 −) during a transition $k_1 + \rightarrow k_2 +$ (or respectively $-k_1 - \rightarrow -k_2 -$), where k_2 also belongs to the region I or III; secondly through the formation in I and III of an excited pair generated by the operator $[(1 - h_k)^{1/2}\overset{+}{b}_k$

$- h_k^{1/2}$]; thirdly by the breaking up in region I or III of a pair $(k_1 +, - k_1 -)$ during a transition $k_1 + \rightarrow k_2 +$, where k_2 belongs to II. In the first case the excitation energy is of the form

$$W_{k_1k_2} - W_0 = \overset{+}{\varepsilon}_{k_2} + \overset{-}{\varepsilon}_{k_1} - (\overset{+}{\varepsilon}_{k_2} + \overset{-}{\varepsilon}_{k_2}) h_{k_2} - (\overset{+}{\varepsilon}_{k_1} + \overset{-}{\varepsilon}_{k_1}) h_{k_1}$$
$$+ \{[h_{k_1}(1 - h_{k_1})]^{1/2} + [h_{k_2}(1 - h_{k_2})]^{1/2}\} V \sum_{\mathrm{I,\,III}} [h_k(1 - h_k)]^{1/2}$$
$$= (\varepsilon_{k_1}^2 + \varepsilon_0^2)^{1/2} + (\varepsilon_{k_2}^2 + \varepsilon_0^2)^{1/2} \tag{14}$$

and its minimum value is $2[\tfrac{1}{2}\mu J)^2 + \epsilon_0^2]^{1/2}$. In the second case we find for the excitation energy

$$W_k - W_0 = (\overset{+}{\varepsilon}_k + \overset{-}{\varepsilon}_k)(1 - 2h_k) \tag{15}$$
$$+ 4\varepsilon_0 [h_k(1 - h_k)]^{1/2} = 2(\varepsilon_k^2 + \varepsilon_0^2)^{1/2}$$

with the same minimum value. In the third case the excitation energy is of the form

$$W_{k_1k_2} - W_0 = \overset{-}{\varepsilon}_{k_1} + \overset{+}{\varepsilon}_{k_2} - (\overset{+}{\varepsilon}_{k_1} + \overset{-}{\varepsilon}_{k_1}) h_{k_1}$$
$$+ 2[h_{k_1}(1 - h_{k_1})]^{1/2} V \sum_{\mathrm{I,\,III}} [h_k(1 - h_k)]^{1/2} = \varepsilon_{k_2} + (\varepsilon_{k_1}^2 + \varepsilon_0^2)^{1/2} \tag{16}$$

with a minimum value $-\tfrac{1}{2}\mu J + [(\tfrac{1}{2}\mu J)^2 + \epsilon_0^2]^{1/2} > 0$ (since the minimum of ϵ_{k_2} is $-\tfrac{1}{2}\mu J$, and the minimum of $|\epsilon_{k_1}|$ is $\tfrac{1}{2}\mu J$) which vanishes only for $\epsilon_0 = 0$. It follows thus that also in the case of a ferromagnetic superconductor ϵ_0 from (11) determines the magnitude of the energy gap for the elementary excitations of the system.

When $T > 0$ one must take into account, apart from the basic pairs and the left-hand singlets, also the excited pairs and right-hand singlets. One must thus take instead of (2) a wave function of the form

$$\Psi' = \prod_{k(0)} [(1 - h_k)^{1/2} + h_k^{1/2} \overset{+}{b}_k] \prod_{k_1(\mathrm{exc})} [(1 - h_{k_1})^{1/2} \overset{+}{b}_{k_1}$$
$$- h_{k_1}^{1/2}] \prod_{k_2(s^-)} \overset{+}{c}_{k}^{-} - \prod_{k_3(s+)} \overset{+}{c}_{k_3}^{+} + \Phi_0, \tag{17}$$

where 0, exc, s^-, and s^+ refer to states occupied respectively by basic pairs, excited pairs, left-hand, and right-hand singlets. We introduce, apart from the probability s_k^-, the probabilities s_k^+ and p_k for finding in the state k respectively a right-hand singlet or an excited pair, and we express them in terms of the distribution functions f_k^+ and f_k^- of fermions with + and − spins:

$$s_k^+ = f_k^+ (1 - f_k^-), \quad s_k^- = f_k^- (1 - f_k^+), \quad p_k = f_k^- f_k^+. \tag{18}$$

(This connection follows from the fact that s_k^+ is the probability that the state with k+ is occupied while the state −k− is vacant; s_k^- the probability that the state with k− is occupied while the state −k+ is vacant, and finally p_k the probability that the state with k+ and the state with −k− are simultaneously occupied.) We get then for the average energy, instead of (8),

$$W = \sum_{k>k_F^+} \overset{+}{\varepsilon}_k [f_k^+ + (1 - f_k^+ - f_k^-) h_k] + \sum_{k>k_F^-} \overset{-}{\varepsilon}_k [f_k^-$$
$$(1 - f_k^+ - f_k^-) h_k] + \sum_{k<k_F^+} |\overset{+}{\varepsilon}_k| [1 - f_k^+ - (1 - f_k^+ - f_k^-) h_k]$$
$$+ \sum_{k<k_F^-} |\overset{-}{\varepsilon}_k| [1 - f_k^- - (1 - f_k^- - f_k^-) h_k]$$
$$- V \sum_{kk'} [h_k(1 - h_k) h_{k'}(1 - h_{k'})]^{1/2} (1 - f_k^+ - f_k^-)(1 - f_{k'}^- - f_{k'}^-). \tag{19}$$

The usual expression for the entropy of fermions

$$S = -\varkappa \sum_k [f_k^+ \ln f_k^+ + (1 - f_k^+) \ln (1 - f_k^+) + f_k^- \ln f_k^-$$
$$+ (1 - f_k^-) \ln (1 - f_k^-)] \tag{20}$$

(where κ is Boltzmann's constant), together with the condition that the free energy, $F = W - TS$, is a minimum

$\partial F/\partial h_k = \partial F/\partial f_k^+ = \partial F/\partial f_k^- = 0$ while $1 - f_k^- - f_k^- \neq 0$, yields

$$f_k^+ = [\exp\{\beta[(\varepsilon_k^2 + \varepsilon_0^2)^{1/2} + \mu J/2]\} + 1]^{-1},$$
$$f_k^- = [\exp\{\beta[(\varepsilon_k^2 + \varepsilon_0^2)^{1/2} - \mu J/2]\} + 1]^{-1},$$
$$h_k = \tfrac{1}{2}[1 - \varepsilon_k/(\varepsilon_k^2 + \varepsilon_0^2)^{1/2}], \tag{21}$$

where $\beta = (\kappa T)^{-1}$ and, in contradistinction to (7),

$$\varepsilon_0 = V \sum_k [h_k(1 - h_k)]^{1/2} (1 - f_k^+ - f_k^-). \tag{22}$$

In III, where $h_k < \tfrac{1}{2}$, the functions f_k^+ and f_k^- describe the distributions of electrons with + and − spin, respectively. In I, where $h_k > \tfrac{1}{2}$, these functions describe the corresponding hole distributions. Since the electron and hole distributions found here are symmetric with respect to $\epsilon_k = 0$, it is necessary to put in II the functions $f_k^- = 1$ and $f_k^+ = 0$, for in that region $1 - f_k^+ - f_k^- = 0$. The summation in (22) and also in (7) extends thus only over the regions I and III.

2. CRITICAL TRANSITION TEMPERATURE

The following equation for ϵ_0 results from (21) and (22)

$$\frac{1}{N(0)V} = \int_{\mu J/2}^{\hbar\omega} \frac{d\varepsilon}{E} \frac{\sinh(E/\varkappa T)}{\cosh(E/\varkappa T) + \cosh(\mu J/2\varkappa T)}, \tag{23}$$

where $E = (\epsilon^2 + \epsilon_0^2)^{1/2}$. Taking it into account that at the critical temperature T_c the quantity ϵ_0 tends to zero, we find from (23) the following equation for T_c

$$\frac{1}{N(0)V} = \int_{\mu J/2}^{\hbar} \frac{d\varepsilon}{\varepsilon} \frac{\sinh(\varepsilon/\varkappa T_c)}{\cosh(\varepsilon/\varkappa T_c) + \cosh(\mu J/2\varkappa T_c)}. \tag{24}$$

In the case $\mu J/2\kappa T_c \ll 1$, we can restrict ourselves to terms of first order in that quantity and we can find from (24) the following equation for T_c

$$1/N(0)V = \ln(1.14\hbar\omega/\varkappa T_c) - \mu J/4\varkappa T_c, \qquad (25)$$

from which, in the same approximation, it follows that

$$\varkappa T_c = 1.14\hbar\omega \exp\{-1/N(0)V\} - \mu J/4, \qquad (26)$$

so that T_c decreases when μJ increases.

When we consider the case of large μJ, with $\mu J/2\kappa T_c \gg 1$, it is convenient to write (24) in the form

$$\frac{1}{N(0)V} = \ln\frac{2\hbar\omega}{\mu J} + 2\sum_{m-1}^{\infty}(-1)^m\left[\mathrm{Ei}\left(-\frac{m\hbar\omega}{\varkappa T_c}\right)\right.$$
$$\left. - \mathrm{Ei}\left(-\frac{m\mu J}{2\varkappa T_c}\right)\right]\cosh\frac{m\mu J}{2\varkappa T_c}, \qquad (27)$$

where Ei is the exponential integral. Using, moreover, an asymptotic expansion for Ei and restricting ourselves to terms linear in $2\kappa T_c/\mu J$ we get easily from (27)

$$\varkappa T_c = \frac{\mu J}{2\ln 2}\left(\ln\frac{2\hbar\omega}{\mu J} - \frac{1}{N(0)V}\right). \qquad (28)$$

It follows from (28) that when $\mu J = 2\hbar\omega \exp\{-1/N(0)V\}$ the critical temperature tends to zero, which agrees with condition (13).

3. SPECIFIC HEAT OF A FERROMAGNETIC SUPERCONDUCTOR

Using (11), we can transform (20) and get for the electron specific heat in the superconducting state the following expression

$$C_{es} = T\frac{\partial S}{\partial T} = -2\varkappa\beta^2 \sum_{k>k_F^-}\left[\left(E_k + \frac{\mu J}{2}\right)\frac{\partial f_k^+}{\partial\beta} + \left(E_k - \frac{\mu J}{2}\right)\frac{\partial f_k^-}{\partial\beta}\right] \qquad (29)$$

Equation (29) can be transformed to

$$C_{es} = 2\varkappa\beta^2 \sum_{k>k_F^-}\left\{f_k^+(1-f_k^+)\left[\left(E_k + \frac{\mu J}{2}\right)^2 + \beta\left(E_k + \frac{\mu J}{2}\right)\right.\right.$$
$$\left.\times\left(\frac{1}{2E_k}\frac{\partial\varepsilon_0^2}{\partial\beta} + \frac{\partial}{\partial\beta}\frac{\mu J}{2}\right)\right] + f_k^-(1-f_k^-)\left[\left(E_k - \frac{\mu J}{2}\right)^2\right.$$
$$\left.\left. + \beta\left(E_k - \frac{\mu J}{2}\right)\times\left(\frac{1}{2E_k}\frac{\partial\varepsilon_0^2}{\partial\beta} - \frac{\partial}{\partial\beta}\frac{\mu J}{2}\right)\right]\right\}. \qquad (30)$$

Considering that at $T = T_c$ the quantity ϵ_0 tends to zero and that the distribution function (21) takes on its normal value

$$f_{k,n}^+ = [\exp\{\beta(\varepsilon_k + \mu J/2)\} + 1]^{-1},$$
$$f_{k,n}^- = [\exp\{\beta(\varepsilon_k - \mu J/2)\} + 1]^{-1}, \qquad (31)$$

one gets easily from (30) an expression for the jump in the specific heat, accurate to terms of order $\mu J/\kappa T_c$,

$$(C_{es} - C_{en})_{T_c} = \varkappa N(0)\,\beta_c^2(1 - \mu J/4\varkappa T_c)\,\partial\varepsilon_0^2/\partial\beta\,|_{T_c}, \qquad (32)$$

where C_{en} is the specific heat of the normal state. In the same approximation, by differentiating (23) with respect to β and using numerical integration, we can find for the quantity $\partial\epsilon_0^2/\partial\beta\,|\,T_c$ the following expression

$$\left.\frac{\partial\varepsilon_0^2}{\partial\beta}\right|_{T_c} = \frac{10.30}{\beta_c^3}\left(1 - 0.07\frac{\mu J}{\varkappa T_c}\right). \qquad (33)$$

From (32) and (33) we get

$$(C_{es} - C_{en})_{T_c} = 10.30\,\varkappa^2 N(0)(1 - 0.32\mu J/\varkappa T_c), \qquad (34)$$

from which it follows that the (s-d)-exchange interaction reduces the magnitude of the jump in the specific heat. Assuming (33) to be valid when $T \lesssim T_c$, we can easily find also the temperature dependence of ϵ_0 for $T \sim T_c$ and $\mu J/\kappa T_c < 1$:

$$\varepsilon_0(T) = 3.209\,\varkappa T_c(1 - T/T_c)^{1/2}(1 - 0.035\,\mu J/\varkappa T_c), \qquad (35)$$

which goes over into the corresponding expression (3.31) of reference 4 when $\mu J \rightarrow 0$.

4. CRITICAL MAGNETIC FIELD

It is well known that the critical magnetic field H_c can be determined by using the relation

$$H_c^2/8\pi = F_n - F_s, \qquad (36)$$

where F_n and F_s are respectively the free energies of the normal and the superconducting states. To evaluate $F_s = W - TS$ we use (21) and (22), changing from a summation over the quasi-momentum to integration over the energy $d\epsilon$, to transform (19) to the form

$$W = 2N(0)\left\{\int_{\mu J/2}^{\hbar\omega}\varepsilon d\varepsilon \int_{\mu J/2}^{\infty}\left[\frac{\mu J}{2}(f_k^+ - f_k^-)\right.\right.$$
$$\left.\left. - \frac{\varepsilon^2}{2E}(e^{\beta(E-\mu J/2)} - 1)(f_k^- + f_k^+)\right]d\varepsilon - \frac{\varepsilon_0^2}{V}\right\}. \qquad (37)$$

Similarly we can use (21), change from summation to integration, integrate by parts, and obtain an expression for the entropy

$$TS = 2N(0)\left\{\int_{\mu J/2}^{\infty}[(\varepsilon^2/E + E)(f_k^+ + f_k^-) + (\mu J/2)(f_k^+ - f_k^-)]\,d\varepsilon\right.$$
$$\left. - \varkappa T(\mu J/2)\ln[2(1 + e^{-\beta\mu J})]\right\}. \qquad (38)$$

We find from (37) and (38), and also from (22) and (21), after some simple calculations

$$F_s = -N(0)\int_{\mu J/2}^{\infty}(2\varepsilon^2 + \varepsilon_0^2)E^{-1}(f_k^+ + f_k^-)\,d\varepsilon + N(0)\{(\hbar\omega)^2[1$$
$$+ \sqrt{1 + (\varepsilon_0/\hbar\omega)^2}] - (\mu J/2)^2[1 - \sqrt{1 + (2\varepsilon_0/\mu J)^2}]\}$$
$$+ N(0)\varkappa T\mu J\ln[2(1 + e^{-\beta\mu J})]. \qquad (39)$$

Substituting (39) and the equation $F_n = -\pi^2 N(0) \times (\kappa T)^2/3$ for the free energy of the normal state into (36), we get

$$H_c^2/8\pi = -N(0)\{(\hbar\omega)^2[1-\sqrt{1+(\varepsilon_0/\hbar\omega)^2}]-(\mu J/2)^2[1-\sqrt{1+(2\varepsilon_0/\mu J)^2}]\}-N(0)\varkappa T\mu J\ln[2(1+e^{-\beta\mu J})]-\frac{\pi^2}{3}N(0)(\varkappa T)^2\left\{1-\frac{3\beta^2}{\pi^2}\int\limits_{\mu J/2}^{\infty}\frac{2\varepsilon^2+\varepsilon_0^2}{E}(f_k^+ + f_k^-)\,d\varepsilon\right\}. \quad (40)$$

From (40) it follows for T = 0 that

$$H_0^2/8\pi = N(0)\{\varepsilon_0^2(0)/2+(\mu J/2)^2[1-\sqrt{1+(2\varepsilon_0(0)/\mu J)^2}]\}, \quad (41)$$

where H_0 is the critical field for T = 0°K. In the case $\mu J \ll 2\hbar\omega \exp\{-1/N(0)V\}$, when $\epsilon_0(0) \gg \mu J$ according to (11), it follows from (41) that

$$H_0 \sim (4\pi N(0))^{1/2}[\varepsilon_0(0)-\mu J/2]. \quad (42)$$

When $\mu J \rightarrow 2\hbar\omega \exp\{-1/N(0)V\}$, when $\epsilon_0(0) \rightarrow 0$ according to (11), it follows from (41) that $H_0 \rightarrow 0$.

Using (41) one can write Eq. (40) in the form

$$H_c^2/8\pi = H_0^2/8\pi - \alpha_1 T - \alpha_2 T^2, \quad (43)$$

$$\alpha_1 = N(0)\varkappa\mu J[\ln 2 + \ln(1+e^{-\beta\mu J})].$$

The (s-d) exchange leads thus not only to a decrease in the critical field but also to the appearance of a linear term $\alpha_1 T$ besides the usual quadratic dependence of H_c^2 on T (reference 4), which is described by the term $\alpha_2 T^2$.

5. POSSIBILITIES OF EXPERIMENTAL VERIFICATION OF THE THEORETICAL DEDUCTIONS; CONCLUSION

We mentioned in the foregoing that the existence of a ferromagnetic superconductor has only been established so far, apparently, in the (Ce, Gd)Ru_2 system.[3] The possibility is, however, not excluded that an improvement in the techniques for obtaining low temperatures will enable us to observe a similar phenomenon also in other systems in which there is a tendency for an intersection of the curves of the critical temperature (T_c) and of the Curie point (Θ) vs. the concentration of one of the components [for instance, in the (Ce, Pr)Ru_2 system[3]]. On the other hand, it is clear from the foregoing theory that in the case of a superconducting ferromagnet the characteristics of the superconducting state differ from the usual ones by a number of features caused by the (s-d)-exchange coupling. An experimental study of these features would thus be not only of interest for the theory of superconductivity, but also for a verification of the (s-d)-exchange model of ferromagnetic metals and above all for an independent estimate of the (s-d)-exchange coupling parameter J. This parameter could, for instance, be determined by a study of such experimentally accessible quantities as the jumps in the specific heat at T_c and in the critical temperatures and the specific heat of the normal state near T_c. One can, apart from this, according to (43), use the experimental determination of the coefficient α_1 in the temperature dependence of H_c^2, and so on. Accordingly, experiments on a detailed study of the superconducting characteristics of the (Ce, Gd)Ru_2 system and also on a search for new superconducting ferromagnets, would be very worthwhile.

From what has been said it follows that the presence of a shift of the Fermi surface, caused by the fermion-magnetization effect, makes the existence of a superconducting state in a ferromagnet possible only when the (s-d)-exchange coupling is sufficiently weak, when condition (13) is satisfied and this shift reduces the energy gap, the critical temperature, the critical magnetic field, and the jump in the specific heat. The effective repulsion of the conduction electrons, induced by the (s-d)-exchange coupling,[7] also acts in the same direction; the appropriate matrix element V_{sd} decreases thus the quantity V in all expressions obtained. Since, however, the quantity V_{sd} is proportional to J^2, one can expect that when the (s-d)-exchange interaction is weak the dominant role will be played by the dependences on μJ obtained in the foregoing, in which this quantity enters linearly.

We note in conclusion that Anderson and Suhl[8] have used the Kittel-Ruderman-Yosida[9,10] representation for the interaction between conduction electrons and spin moments to conclude that a special state of spin-atomic ordered "cryptoferromagnetism" could possibly exist in the form of a domain structure with zero resultant moment. The scheme for calculations given in our paper could be generalized also for the case of dilute solutions. We shall present a detailed analysis of that case in another paper.

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INVESTIGATION OF THE (n, p) REACTION IN Li^6

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We have derived the angular distribution of protons produced from the (n, p) reaction in Li^6, taking into account the neutron-proton correlation in the nucleus.

1. It is known that the theory of (n, p) reactions, based on the assumption of direct interaction of the incident nucleon with the nucleus, leads (in the plane wave approximation) to an angular distribution of scattered particles which is in agreement with experimental data only at relatively small angles, up to 70 – 90°.[1,2] For large scattering angles, the theory gives a rapidly oscillating distribution curve which approaches zero with increasing scattering angle, whereas the experimental curve for these angles is a smooth and, in the majority of cases, monotonically decreasing curve.

It is interesting that for the case of Li^6, as shown by the work of Frye,[3] instead of a drop in the distribution at large scattering angles, one observes a rise which begins after reaching the minimum around 70°. As seen from the curve given by Frye, the effective cross section for the (n, p) reaction at 120° has the same order of magnitude as the cross section near zero, where it takes on its maximum value.

In the present paper we investigate the (n, p) reaction on Li^6 on the basis of the α-deuteron model of this nucleus which was developed by the authors.[4,5] We may expect that the study of this reaction on the basis of the α-deuteron model of Li^6 will lead to better agreement with experiment. Assuming that the neutron and proton which are outside the closed shell in Li^6 form a bound state, we must conclude that the ejection of the proton may be caused by the interaction of the incident neutron, not only directly with the ejected proton, as is assumed in existing theories of nuclear reactions, but also with the neutron which is bound to it.

It is clear that the effective cross section for the process of ejection of a proton as a result of interaction of the incident neutron directly with this proton will give a forward maximum in the angular distribution, whereas the interaction with the bound neutron, because of exchange effects, will lead to a rise in the cross section for this process at large scattering angles, in agreement with the experimental data.

2. The interaction energy of the incident neutron with the nucleus we assume to have the form

$$U = A\{\delta(\mathbf{r}_3 - \mathbf{r}_1) + \delta(\mathbf{r}_3 - \mathbf{r}_2)\}, \tag{1}$$

where $\mathbf{r}_1$ and $\mathbf{r}_2$ are the radius vectors of the proton and neutron in the Li^6 nucleus, and $\mathbf{r}_3$ is the radius vector of the incident neutron.

The wave functions of the system in the initial and final states, antisymmetrized with respect to the neutrons, have the following forms:

$$\psi_i = \frac{1}{\sqrt{2}}\Big\{e^{i\mathbf{k}_3\mathbf{r}_3}\chi_{\mu_3}(3)\,\psi_d(|\mathbf{r}_1 - \mathbf{r}_2|)\,\psi_1\Big(\frac{\mathbf{r}_1+\mathbf{r}_2}{2}\Big)\chi_1(1,2) - e^{i\mathbf{k}_3\mathbf{r}_2}\chi_{\mu_3}(2)\,\psi_d(|\mathbf{r}_1 - \mathbf{r}_3|)\,\psi_1\Big(\frac{\mathbf{r}_1+\mathbf{r}_3}{2}\Big)\chi_1(1,3)\Big\}, \tag{2}$$

$$\psi_f = e^{i\mathbf{k}_1\mathbf{r}_1}\chi_{\mu_1}(1)\,\psi_2\Big(\frac{\mathbf{r}_2+\mathbf{r}_3}{2}\Big)\psi_{nn}(|\mathbf{r}_2 - \mathbf{r}_3|)\,\chi_0(2,3), \tag{3}$$

where $\mathbf{k}_3$ and $\mathbf{k}_1$ are the wave vectors of the incident and emerging particles in the center of mass system; $\psi_d(|\mathbf{r}_1 - \mathbf{r}_2|)$ is the internal wave function of the neutron-proton pair in the Li^6 nucleus; ψ_1 is the wave function for the relative motion of this pair and the α particle; ψ_2 is the wave function for the relative motion of the center of the neutron-neutron pair in the He^6 nucleus at the end of the process (we treat the He^6 nucleus as a system of an α particle and two neutrons which are not bound to one another); $\psi_{nn}(|\mathbf{r}_1 - \mathbf{r}_3|)$ is the internal wave function of this pair of neutrons; $\chi_1(1, 2)$ and $\chi_0(2, 3)$ are the spin wave functions for the triplet and singlet states of a pair of nucleons; $\chi_\mu(3)$, $\chi_\mu(1)$, and $\chi_\mu(2)$ are the spin wave functions of the individual nucleons.

Assuming that both the relative motion of the center of the (nn) pair and the α particle in the He^6 nucleus, as well as of the (np) pair in Li^6 are described by an S state, we obtain for the square of the scattering amplitude

$$|f(\theta)|^2 \sim \{I_1(q) + \psi_d(0)\, I_2(q, Q) + 2\psi^*_{nn}(0)\, I_3(q, Q)\}; \tag{4}$$

$$I_1(q) = \int e^{i\mathbf{qR}}\psi^*_2(\mathbf{R})\,\psi_1(\mathbf{R})\,d\mathbf{R}\int \psi^*_{nn}(\mathbf{r})\,\psi_d(\mathbf{r})\,e^{i\mathbf{qr}/2}\,d\mathbf{r},$$

$$I_2(q, Q) = \int e^{i\mathbf{qR}}\psi^*_2(\mathbf{R})\,d\mathbf{R}\int e^{-i\mathbf{Qr}/2}\psi^*_{nn}(\mathbf{r})\,\psi_1\left(\mathbf{R} + \frac{\mathbf{r}}{2}\right)d\mathbf{r}, \tag{5}$$

$$I_3(q, Q) = \int e^{i\mathbf{qR}}\psi_1(\mathbf{R})\,d\mathbf{R}\int e^{-i\mathbf{Qr}/2}\psi_d(\mathbf{r})\,\psi^*_2\left(\mathbf{R} - \frac{\mathbf{r}}{2}\right)d\mathbf{r}.$$

where

$$\mathbf{q} = \mathbf{k}_3 - \mathbf{k}_1, \quad \mathbf{Q} = \mathbf{k}_3 + \mathbf{k}_1. \tag{6}$$

3. We use oscillator wave functions for ψ_1 and ψ_2, i.e.,

$$\psi_1(\mathbf{R}) = \psi_2(\mathbf{R}) = B\exp\left\{-\frac{1}{4}(R/r_0)^2\right\}, \tag{7}$$

where B is a normalization factor, r_0 is a parameter associated with the frequency of the oscillator potential. We choose for the wave function of the (np) pair in the Li^6 nucleus in the initial state the deuteron wave function

$$\psi_d = N\,(e^{-\alpha r} - e^{-\beta r})/r, \tag{8}$$

where N is a normalization factor, $\beta = 1.63 \times 10^{13}$ cm^{-1}, $\alpha = 0.23 \times 10^{13}\,cm^{-1}$.

In choosing the form of the wave function ψ_{nn}, we can be guided by the following considerations: We know that the binding energy of the deuteron to the α particle in Li^6 is equal, according to data on mass defects, to 1.47 Mev, which differs little from the binding energy of the pair of neutrons (which are not bound to one another) to the α particle in He^6, which is equal to 1.27 Mev. From this it follows that the energy associated with the relative motion within the (nn) pair is positive and close to zero. This fact enables us to regard the neutrons of the (nn) pair in He^6 as being correlated in the sense of Levinger,[6] so that we may ascribe zero relative energy to them. Then the function ψ_{nn} can be chosen to have the form

$$\psi_{nn} = \frac{D}{r}\left\{\frac{\sin(kr + \delta_s)}{\sin\delta_s} - e^{-\beta_s r}\right\}_{k\to 0} \approx \frac{D}{r}\{1 - e^{-\beta_s r}\}, \tag{9}$$

$$\beta_s = 1.185\cdot 10^{13}\,\mathrm{cm}^{-1}. \tag{10}$$

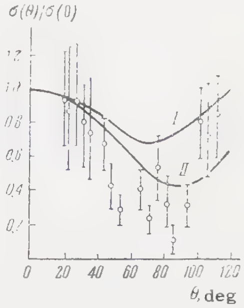
In calculating the integrals I_2 and I_3 in formula (4), we assume that the main contributions come from small R, and we set $\exp(\mathbf{qR}) \approx 1$. As a result, we obtain for the square of the scattering amplitude

$$|f(\theta)|^2 \sim e^{-(qr_0)^2}\left\{\frac{F(q)}{qr_0} + \frac{2(\beta - \alpha)\,r_0 C(Q)}{Qr_0} + \frac{4\beta_s r_0}{4(\alpha r_0)^2 + (Qr_0)^2} - \frac{2(\beta - \alpha)\,r_0 + 4\beta_s r_0}{4(\beta_s r_0)^2 + (Qr_0)^2}\right\}; \tag{11}$$

$$F(q) = \tan^{-1}\frac{2(\beta - \alpha)\,q}{q^2 - 4\alpha\beta} + \tan^{-1}\frac{2(\alpha - \beta)\,q}{q^2 + 4(\alpha - \beta)(\beta - \beta_s)}, \tag{12}$$

$$C(Q) = \int_0^\infty \sin(Qr_0x)\exp\left\{-\left(\frac{x}{2}\right)^2\right\}dx. \tag{13}$$

As we see, the square amplitude contains a term $e^{-(qr_0)^2}F(q)/qr_0$ which corresponds to an ordinary scattering process not associated with exchange. This is a decreasing function of the scattering angle and consequently describes the process of direct ejection, which has a maximum in the forward direction. As for the other three terms, they are caused by (n, p) correlations which we have taken into account via the exchange effects. As one sees easily from formula (11), for small values of $r_0 \approx 1 \times 10^{-13}$ cm the exchange part of the amplitude increases with increasing scattering angle, and leads to a corresponding rise in the effective cross section.



The angular distributions obtained from formula (11) for an incident neutron energy of 14 Mev are shown in the figure. Curve I corresponds to a value of the parameter for the oscillator potential $r_0 \approx 1 \times 10^{-13}$ cm, and curve II to the value $r_0 = 1.2 \times 10^{-13}$ cm; the points with the indicated limits of error show the experimental data. As we see from the figure, the theoretical curves correctly reproduce the dependence of the effective cross section on scattering angle, although it should be emphasized that their form depends strongly on the value of the oscillator parameter r_0. Somewhat better agreement with experiment is obtained for $r_0 = 1.2 \times 10^{-13}$ cm.

As we see, taking account of the fact that the incident neutron can interact not only with the proton which is ejected from the nucleus, but also with the neutron in the nucleus which is correlated to the proton, we get a definite contribution to the total cross section for the (n, p) reaction in Li^6, which leads to an increase in the cross section at large scattering angles. We may expect that including this effect will have an appreciable result for the case of (n, p) reactions on heavy nuclei also.

In conclusion, it is our pleasant duty to express our gratitude to V. I. Mamasakhlisov for continued interest in our work and for valuable discussion.

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SOVIET PHYSICS JETP VOLUME 12, NUMBER 2 FEBRUARY, 1961

BUILD UP OF ION ACOUSTIC VIBRATIONS IN AN ANISOTROPIC PLASMA

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The vibrations of an anisotropic plasma located in a magnetic field are investigated for small values of $\beta = 8\pi n T_\perp/H^2$. It is found that the ion acoustic waves may be unstable in this case.

USUALLY in the investigation of the stability of plasma, it is assumed that the mean Larmor radius of the ions r is small in comparison with the characteristic dimensions of the problem, for example the wavelength λ, inasmuch as this materially simplifies the problem. The stability of an anisotropic plasma has been investigated by us for $\lambda \sim r$ and $\lambda \ll r$. It is assumed that the ions have a Maxwellian distribution with different temperatures across and along the magnetic field. This is one of the simplest cases of a non-equilibrium distribution.

The effect of the temperature anisotropy on the build up of transverse vibrations in a homogeneous plasma under the assumption $r/\lambda \to 0$ was investigated by Vedenov and Sagdeev[1] and by Sagdeev and Shafranov.[2] They have shown that the plasma becomes more stable under a decrease in pressure. In fact, if the plasma pressure is much less than the pressure of the magnetic field, i.e., $\beta = 8\pi n T_\perp/H^2 \ll 1$, then the thermal velocity of the ions v_T is much smaller than the Alfven velocity v_A and consequently only a small fraction of the particles have velocities on the order of the phase velocity of the wave. But the vibrations can set into motion only particles which resonate with the wave, since only these particles effectively exchange their energy with it. Therefore, for $\beta = v_T^2/v_A^2 \ll 1$, the plasma will be practically stable.

The problem as to whether this is maintained for much smaller wavelengths, with $\lambda \ll r$, is of interest.

The dispersion equation for the frequency of the vibrations of a homogeneous plasma[5] contains components of the dielectric-constant tensor, obtained by Shafranov and Sagdeev[2] for an anisotropic plasma in the form of infinite sums. In the investigation of transverse vibrations it was assumed for simplicity that the electrons were cold, $T_e = 0$, the wavelength was much smaller than the mean Larmor radius of the ions, $kr \gg 1$ (k is the wave number), and the direction of propagation of the wave was perpendicular to the magnetic field, $kr \cos\theta \ll 1$. Under these assumptions, the expressions for the components of the tensor can be summed. Investigation shows that for small values of β the vibrations are stable. Thus, for $r/\lambda \gg 1$, just as for $r/\lambda \ll 1$, decrease of β has a stabilizing influence on the transverse vibrations.

Therefore, if $\beta \ll 1$, great interest attaches to the longitudinal vibrations. Harris[4] discovered instability of the plasma in relation to the Langmuir electron oscillations for certain simple types of non equilibrium distributions. We consider ion acoustic vibrations, which are also longitudinal. In a plasma with $T_e \gg T_i$, the phase velocity of these vibrations is determined by the temperature of the electrons; therefore, a change in β cannot have a substantial effect on their development. Consequently, it is natural to assume that just the oscillations may be unstable for small β.

The dispersion equation for the ion sound reduces to the equation

$$A = A_i + A_e = \varepsilon_{11}\sin^2\theta + 2\varepsilon_{13}\sin\theta\cos\theta + \varepsilon_{33}\cos^2\theta = 0,$$

$$\varepsilon_{ik} = \overset{i}{\varepsilon}_{ik} + \overset{e}{\varepsilon}_{ik}, \quad \theta = \widehat{\mathbf{kH}},$$

where

$$A_i = \frac{\varepsilon_0}{b}\sum_{n=-\infty}^{\infty}\left\{\frac{aX}{2\sqrt{d}}Y_n + \frac{n}{2\sqrt{d}}(1-X)Y_n - X\right\}e^{-b}I_n(b)$$

is the contribution of the ions to the dispersion equation. Here we have introduced the notation

$$a = \omega/\Omega, \quad b = T_\perp k_\perp^2/M\Omega^2, \quad d = T_\parallel k_\parallel^2/2M\Omega^2,$$

$$\varepsilon_0 = 4\pi e^2 n/M\Omega^2, \qquad \Omega = eH/Mc, \quad X = T_\perp/T_\parallel,$$

$$Z_n = (a-n)/2\sqrt{d},$$

$$Y_n = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}e^{-t^2}\frac{dt}{Z_n - t} = \begin{cases} -i\sqrt{\pi} + Z_n + \dots, & Z_n \ll 1 \\ -i\sqrt{\pi}\exp(-Z_n^2) + Z_n^{-1} + \frac{1}{2}Z_n^{-3}, & Z_n \gg 1 \end{cases}.$$

On the boundary of the region of instability,

$$\operatorname{Im} a = 0,$$

$$\operatorname{Im} A_i = \frac{\varepsilon_0}{b}\sqrt{\frac{\pi}{2d}}\sum_{n=-\infty}^{\infty}[(a-n)X + n]e^{-Z_n^2}I_n(b)e^{-b}.$$

For $T_\perp = T_\parallel$, when the vibrations are damped, $\operatorname{Im} A_i > 0$. Thus the instability is possible in those regions of the values of the parameters a, b, d, and X where Im $A_i < 0$.

It follows from the expression for the imaginary part that the zero term always gives a positive contribution (Landau damping). The build up of the vibrations is brought about by the remaining terms. It is not difficult to see that Im A_i can be negative only when $X > 1$ and $n + 1 > a > n + \frac{1}{2}$ (it is taken into account here that the contributions of the n-th and (n + 1)-th exponents are the largest). The most dangerous in the sense of instability is the region $1 > a > \frac{1}{2}$, which is therefore investigated in the present work.

The contribution of the electrons can be written down similarly. However, since $m \ll M$, it reduces to

$$A_e = \varepsilon_0 X Z_0 b^{-1} \{\mu + i\sqrt{\pi\mu^3 m/M}\}, \quad \mu = T_{\parallel}/T_{\parallel e}.$$

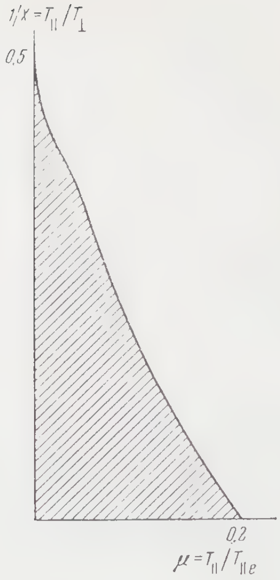
It is found that in order to find the characteristic part of the instability region (μ, $X \sim 1$) it is necessary to consider the values b, $d \sim 1$. The equation is simplified here because only the zeroth and first terms need be considered, since I_n and Y_n decrease with increasing n. Subsequent checking shows that such an approximation is correct. Thus, on the boundary of the instability, Im a = 0, we have

$$(Z_0 \operatorname{Re} Y_0 - 1) I_0 e^{-b} + (|Z_1||\operatorname{Re} Y_1| - 1) I_1 e^{-b} - X^{-1}(Z_0 + |Z_1|)|\operatorname{Re} Y_1| I_1 e^{-b} = \mu$$

$$Z_0 e^{-Z_0^2} I_0 e^{-b} - |Z_1| e^{-Z_1^2} I_1 e^{-b} + X^{-1}(Z_0 + |Z_1|) e^{-Z_1^2} I_1 e^{-b} + Z_0 \sqrt{\mu^3 m/M} = 0.$$

In the present research we attempted to compute the maximum values of $\mu(X^{-1})$ (in Z_0, Z_1, b) satisfying the system. The region of instability is shown in the drawing (shaded). The maximum value of μ for which instability is possible is $\mu_{\max} \approx 0.2$, while the minimum occurs at $X_{\min} \approx 2$. Close to $\mu = 0$, the role of the electrons increases, since the contribution of the ions to the imaginary part is decreased.

It is not difficult to see that the build up of ion acoustic vibrations in an anisotropic nonisothermal plasma comes about as a result of cyclotron resonance. Such resonance is most effective when the wavelength λ is of the order r, and vanishes as $r/\lambda \rightarrow 0$. Here the vibration frequency is of the order of the cyclotron frequency of the ions.



We have thus shown that an anisotropic, nonisothermal plasma, μ, $X \neq 1$, is unstable under ion sound vibrations, while, in contrast to the instability for the Alfven branch, these vibrations are also sent into motion for.

$$\beta = P_{\text{plasma}}/P_{\text{mag. field}} \lll 1.$$

Consequently, the ion acoustic vibrations can be set in motion even in the case of relatively large magnetic fields. It should be noted that in the characteristic parts of the instability region the increment can be large, reaching values of the order of the cyclotron frequency; therefore, the ion acoustic vibrations develop first.

In conclusion, I express my deep gratitude to B. B. Kadomtsev for help and guidance.

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SOVIET PHYSICS JETP VOLUME 12, NUMBER 2 FEBRUARY, 1961

A NEW RESONANCE EFFECT IN METALS AT HIGH FREQUENCIES

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The current density **j** and the electric field intensity **E** in metals is investigated at frequencies exceeding those for which cyclotron resonance was previously studied.[1] It is shown that, in the case of resonance, **j** and **E** on the central cross section of the Fermi surface (when the latter is not an ellipsoid) vary with the depth y in a very peculiar way. At not too great a depth ($y \ll d^2/\delta_0$, where d is the Larmor orbit diameter and δ_0 is the skin depth, $d \gg \delta_0$) the field strength and the current have sharp maxima not only at the surface of the metal but also for $y \approx d, 2d, 3d, \ldots$, (Fig. 2). For $\gtrsim d^2/\delta_0$, the current and the field oscillate at distances of the order of d with a damping depth of the order of d^2/δ_0. In this connection, some new effects are predicted, such as discontinuities in the resonance impedance (with resonance conditions maintained) and discontinuous disappearance of resonance at the harmonics in plates of thickness $D > d$: 1) when the frequency of the field increases and 2) when a constant magnetic field is rotated in the plane of the film. Other new effects are selective transparency of films at resonance, and an electronic "echo" similar to the spin "echo." A study of the impedance of plates permits one to plot the Fermi surface directly. It is also shown that cyclotron resonance rather than diamagnetic resonance takes place in a number of semiconductors and poor metals.

1. PHYSICAL REASON FOR SPIKES IN FIELD AND CURRENT

It is well known that a variable electromagnetic field in a metal is damped the more rapidly the higher the frequency of the field. However, we shall show that there exists a very important special case in which, in a metal at a depth much greater (by two orders of magnitude) than the usual skin depth, there exist sharp maxima of the value of the field. The field in these spikes is of the same order of magnitude as on the surface of the metal.

Inasmuch as "spikes" of the field much deeper in the metal than the skin depth have not been encountered earlier, so far as we know, we shall first study the physical peculiarities of the given case.

Let us first consider the motion of electrons in a metal along one of the orbits (orbit 1, Fig. 1) passing through the skin layer close to the surface of the metal, where the electric field is not small (the constant magnetic field **H** is perpendicular to the plane of the drawing). In a layer of the order of δ, the electrons along a path of length $\sqrt{r\delta}$ receive a directed velocity and produce a current J and a current density **j**.

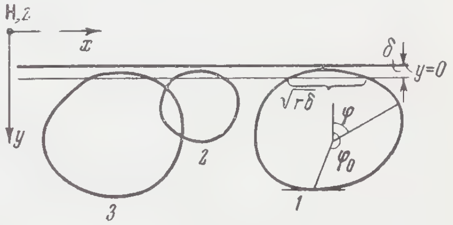

FIG. 1

Downward motion along the orbit changes, in the first place, the electron velocity parallel to the plane of the metal (only then does current flow), causing current to change in proportion to cos φ; in the second place, the electrons will "disperse" throughout the depth, being contained (for φ w $\sqrt{\delta/r}$) not in a layer δ but in a layer $\sqrt{r\delta}$ sin φ.

Thus the current density brought about by electrons of a given orbit is shown to be of the order J cot $\varphi/\sqrt{\delta r}$, i.e., decreasing rapidly with depth; for $\varphi \sim 1$, it will be $\sqrt{r/\delta}$ less than in the layer δ. A value of **j** of the same order as on the surface ($\mathbf{j} \sim J/\delta$) is evidently achieved when $\varphi \sim \sqrt{\delta/r}$, i.e., at a distance of the order of δ. At a depth $y > r$, the current density changes sign, remaining small in absolute value in comparison with J/δ until the angle φ come close enough to φ_0 so that

$| \varphi_0 - \varphi | \lesssim \sqrt{\delta/r}$. At a depth d, the current density would differ in this case from the current density on the surface only in its sign.

Such a current density would create an electric field which would cause directed motion (along the surface) of other electrons moving in the body of the metal, and the appearance of a spike of current and field at $y = 2d$, etc. As a result, it would be necessary to determine this self-consistent system of currents and fields. However, from earlier considerations given above, it is clear that the graph plotted in Fig. 2 is completely natural.

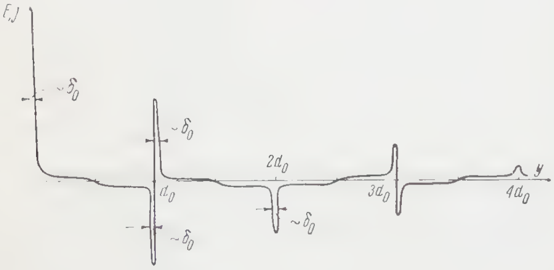

FIG. 2

So far we have discussed one of the orbits penetrating through the skin layer, an orbit with a given radius r. It is clear that all orbits of a given radius penetrating the skin layer (the scatter of the coordinates of their centers is obviously of the order δ) do not change the character of the picture just described.

Another picture arises in the presence of orbits with different radii (for example, orbits 2 or 3 in Fig. 1), corresponding to different cross sections of the Fermi surface [we recall, for example, that for free electrons $r = p_\perp c/eH = c\sqrt{2m\epsilon - p_z^2}/eH$, $0 \le p_z \le \sqrt{2m\epsilon}$, where **p** is the quasi-momentum and r changes from 0 to $c\sqrt{2m\epsilon}/eH$]. The scatter in the radii evidently leads to the result that only a small part of the electrons (of order δ/r) is "collected" at an arbitrary depth in a layer of order δ, and the field "pulled along" into the depth will naturally fall off extremely rapidly in subsequent "links" when $y > d$.

In the case in which the cyclotron frequency Ω of rotation around an orbit does not depend on the cross section (i.e., when the Fermi surface is an ellipsoid), this scatter in the radii cannot be eliminated. But if the cyclotron resonance depends on the cross section (i.e., on p_z, where z is the direction of the constant magnetic field), then it is possible to make use of the cyclotron resonance to eliminate a significant amount of the scatter in the radii; in this resonance (see reference 1), only electrons close to the extremal cyclotron frequencies, with scatter in p_z of the order of Δp_z, take part:

$$|\omega - \Omega(p_z)| \sim 1/\tau, \quad \Delta p_z \sim p_0/\sqrt{\omega\tau}$$

(p_0 is of the order of the limiting Fermi momentum, τ is the time of free flight of the electrons, and ω is the frequency of the rf field); for the existence of resonance it is necessary in any case that $\omega\tau \gg 1$.

The scatter in the radii possible for a given ωτ is quite different for the central cross section where, as is known from symmetry considerations, Ω and d have extrema simultaneously, and for other cross sections, corresponding to an extremum of Ω, where there is no reason to expect an extremum of d.

In the first case, when $d'(0) = 0$, $\Delta d = (1/2) d''(0)(\Delta p_z)^2 \sim d/\omega\tau$, for the existence of the structure of Fig. 2 it is necessary (and, as will be seen below, sufficient) that $\Delta d \sim \delta_0$, i.e.,

$$\omega\tau \gtrsim r/\delta_0 \qquad (r \sim v/\omega, \quad \delta_0 \sim c/\omega_0) \tag{1.1}$$

(v is the velocity of electron, c is the velocity of light, ω_0 is the plasma frequency, $\omega_0 \sim \epsilon_0/\hbar$, ϵ_0 is the limiting Fermi energy). This is observed at frequencies much higher than those studied in reference 1.

The condition (1.1) can be rewritten in the form

$$\omega \gtrsim v/\sqrt{l\delta_0}, \qquad H = m^* c\omega/e \quad (l \sim v\tau), \tag{1.2}$$

which for $l \sim 10^{-1}$ cm, $\delta_0 \sim 10^{-5}$ cm, $v \sim 10^8$ cm/sec corresponds to a wavelength of the order of 1 cm and $H \sim 10^4$ oe, i.e., values feasible in experiments on cyclotron resonance.

In the remaining cases (extremal Ω corresponding to non-central cross sections) we have $\Delta d = d'\Delta p_z \sim d/\sqrt{\omega\tau}$, and the condition for the existence of the structure shown in Fig. 2 is

$$\omega\tau \gtrsim (r/\delta_0)^2.$$

It is easy to prove that this conclusion is valid also for elliptical reference points (where $\mathbf{v} \parallel \mathbf{H}$), in which the role of p_z is played by the angle φ, measured along the arc $\epsilon = \epsilon_0$, $v_y = 0$ (see reference 1). However, in this case the condition $\Delta d \sim \delta_0$ corresponds to the simultaneous disappearance of the strong cyclotron resonance, since it calls for $\Delta d \gg \delta_0$ near the reference points.

We note that the stronger inequality $\omega\tau \gg (r/\delta_0)^2$ cannot be achieved in general. In fact, for cyclotron resonance (when even a small deflection of the electron by collision with a phonon takes the electron out of the skin layer and out of the "resonance" central cross section and is therefore very important), the number of electron-phonon collisions is

$$\nu_{\text{eff}} = 1/\tau_{\text{eff}} \sim \frac{k\Theta}{\hbar}\left(\frac{\hbar\omega + kT}{k\Theta}\right)^3 \gtrsim \frac{k\Theta}{\hbar}\left(\frac{\hbar\omega}{k\Theta}\right)^3$$

(Θ is the Debye temperature). Therefore, since $\tau \lesssim \tau_{\text{eff}}$, we have

$$\omega\tau(\delta_0/r)^2 \lesssim \left(\frac{k\Theta}{\hbar} \Big/ \omega_0\frac{v}{c}\right)^2 \sim \left(\frac{k\Theta}{\varepsilon_0}\frac{c}{v}\right)^2 \sim 1,$$

because in practice $\epsilon_0 v/c \sim k\Theta$ for all metals.

Thus, "spikes" of current and field are possible for central cross sections if

$$\omega\tau \sim (r/\delta_0)^2, \quad \hbar\omega \gtrsim kT, \quad l \sim l_{\text{eff}} \gtrsim (v\hbar/k\Theta)(\Theta/T)^3.$$

For $l \sim 10^{-1}$ cm, this corresponds to $T \gtrsim 10°$ K, $\lambda \lesssim 1$ mm, $H \sim 10^5$ oe, which is (in the case of λ and H) at the limit of today's experimental capabilities.

If conditions (1.3) are fulfilled, an important effect is exerted on the resonance by the electric field perpendicular to the surface of the metal, and also by Fermi-liquid effects which, however, do not complicate the picture of Fig. 2. This is connected with the circumstance that if condition (1.3) is satisfied the effective conductivity is

$$\sigma_{\text{eff}} \sim \sigma(\delta/r)\sqrt{\omega\tau} \sim \sigma$$

(δ/r takes into account the "ineffectiveness" of electrons outside the skin layer, and $\sqrt{\omega\tau}$ is the number of rotations at resonance averaged over the "essential" orbits).

A subsequent paper will be devoted to consideration of the case in which the condition (1.3) is satisfied, which makes it possible to obtain some information on the correlation function in a Fermi liquid, but which is much more complicated. In the present paper, for simplicity of presentation, only resonance of the central cross section is considered, with

$$(r/\delta_0)^2 \gg \omega\tau \gtrsim r/\delta_0.$$

Naturally, the main interest attaches to the case in which the stronger inequality on the right also holds.

We note in passing the following circumstance, which does not apply directly to the theme of this paper but which is worthy of mention. For a square law of dispersion, a sharp increase (by a factor of $\omega\tau$) in the conductivity at resonance can lead at sufficiently large $\omega\tau$ to the result that the anomalous skin effect will correspond to the conductivity $\sigma\omega\tau$ in a poor metal or even in an alloyed semiconductor, and the diamagnetic resonance will be replaced by cyclotron resonance. For this case it is necessary that

$$\delta^2 \sim c^2m/2\pi ne^2\omega\tau \sim \delta_0^2/\omega\tau \ll r^2 \sim v^2/\omega^2,$$

that is,

$$\delta_0^2 \ll vl/\omega.$$

For bismuth, for example, at $\omega \sim 10^{11}$ sec^{-1} and $l \sim 10^{-1}$ cm, this inequality is very well satisfied.

It is possible that the divergence of experimental results with ordinary theory is explained by precisely this circumstance (see, for example, the work of Lax[2]), inasmuch as one must treat the experimental data according to the theory of cyclotron resonance (Lax also pointed out the fact that the data correspond to the theory of cyclotron resonance).

2. DETERMINATION OF THE FIELD INTENSITY IN THE METAL

It is well known that in the one-dimensional case the Maxwell equations (with the obvious neglect of the displacement current) reduce to the form

$$E''_\alpha = (4\pi i\omega/c^2)\,j_\alpha, \quad \alpha = x, z; \qquad j_y = 0, \tag{2.1}$$

where the relation between **j** and **E** must be found from the kinetic equation for the electron distribution function in the metal.

It can be shown (for example, in a fashion similar to what was done in reference 1) that when $\omega\tau \ll (r/\delta_0)^2$ the equations separate: the terms containing E_y, which does almost no work, are small in the expression for j_α (and we can formally set $E_y = 0$ in j_α), while $j_y = 0$ is an equation that determines only E_y, which does not present much interest to us. [In a paper devoted to the case $\omega\tau \sim (r/\delta_0)^2$, a method will be given for separating the equations in the general case.]

A general formula for j_α will be found in reference 1; however, taking it into account that a relaxation time can be introduced for the anomalous skin effect,[1] j_α can be determined much more simply (see the work of Chambers[3]). Evidently,

$$\mathbf{j}e^{i\omega t_1} = \frac{2e}{h^3}\int \mathbf{v}n\,dp_x dp_y dp_z = \frac{2e}{h^3}\frac{eH}{c}\int \mathbf{v}n\,d\varepsilon\, dp_z dt, \tag{2.2}$$

where n is the distribution function, t is the time for rotation of the electron about the orbit (which is determined by the equation $d\mathbf{p}/dt = \dot{\mathbf{p}} = \frac{e}{c}\mathbf{v}\times\mathbf{H}$, $\mathbf{v} = \partial\epsilon/\partial\mathbf{p}$), and t_1 is the "ordinary" time. The quantities ϵ and p_z determine the position of the orbit (p_z) on the Fermi surface $\epsilon(\mathbf{p}) = \epsilon$. The quantity t can be interpreted in this sense, that it determines the location of the center of the orbit, in this case taking the integration over t from 0 to $T = 2\pi/\Omega = 2\pi m^*c/eH$ (m^* is the effective mass, Ω is the cyclotron frequency) to mean integration over the centers of all orbits passing through the given point y.

In fact, let t be the time of transit of the electron from the top of the trajectory to the given point; then the statement that the electron has arrived at the point after a time t means that the coordinate of the center of the orbit is $y - r(t)$ (see Fig. 3; the determination of the center of the orbit is clear from the drawing), with

$$r(t) = \int^{t} v_y dt = \frac{cp_x(t)}{eH} + \text{const.}$$

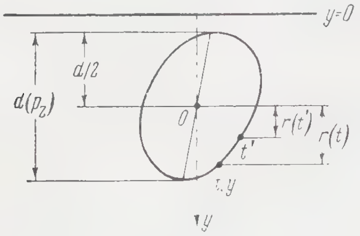

FIG. 3

Such an approach makes immediately obvious the consideration of the boundary condition in the region close to resonance ($|\omega - q\Omega| \ll \omega$, $q = 1, 2, \ldots$), where the only electrons of importance are those which do not collide with the surface. It is clear that an electron does not collide with the surface if the top of the orbit lies inside the metal, i.e., if

$$y - r(t) - r_0 > 0, \quad r_0 = d/2.$$

Therefore, in order to take into account only electrons that do not collide with the surface, it is necessary to add the factor $s[y - r(t) - r_0]$ under the integral sign in (2.2), with $s(w) = 1$ for $w > 0$ and $s(w) = 0$ for $w < 0$.

Now let us determine n. Inasmuch as the number of electrons remains unchanged, the distribution function changes only as the result of change in energy of the electrons, so that

$$n(y, t_1) = n_0(\varepsilon - \Delta\varepsilon) = n_0(\varepsilon) - \frac{\partial n_0}{\partial \varepsilon}\Delta\varepsilon(y, t_1), \tag{2.3}$$

where $\Delta\epsilon(y, t_1)$ is the energy acquired by an electron incident at the point y at the time t_1 ($\dot{\epsilon} = e\mathbf{v}\cdot\mathbf{E}$). Determining $\Delta\epsilon$ and substituting it in (2.3), and then substituting (2.3) in (2.2), we obtain

$$j_\alpha(y) e^{i\omega t_1} = \frac{2e^2}{h^3}\frac{eH}{c} e^{i\omega t_1} \int v_\alpha(t) e^{-i\omega t - t/\tau} s(y - r(t) - r_0) \frac{\partial n_0}{\partial \varepsilon} d\varepsilon dp_z dt \times \int_{-\infty}^{t} \mathbf{v}(t')\mathbf{E}(y - r(t) + r(t')) e^{i\omega t' + t'/\tau} dt', \tag{2.4}$$

which gives the relation between **j** and **E**.

Now, taking into account the periodicity of the velocities and $r(t)$ in t with period T, we transform $\int_{-\infty}^{t}$ into $\int_{t-T}^{t}$ and keep only the principal term in $1/\omega\tau \ll 1$. Noting that $\partial n_0/\partial\epsilon \approx -\delta(\epsilon - \epsilon_0)$, we obtain the following expression for j near resonance ($|\omega - q\Omega| \ll \omega$).

$$j_\alpha(y) = \frac{2e^3H}{h^3c}\int [1 - \exp(-2\pi i\omega/\Omega - 2\pi/\Omega\tau)]^{-1} dp_z \times \int_0^T v_\alpha(t) e^{-i\omega t} s(y - r(t) - r_0) dt \times \int_0^T v_\beta(t') E_\beta(y - r(t) + r(t')) e^{i\omega t'} dt',$$

$$v_\beta E_\beta \equiv v_x E_x + v_z E_z, \tag{2.5}$$

where $[1 - \exp(-2\pi i\omega/\Omega - 2\pi/\Omega\tau)]^{-1}$ evidently gives a large number of rotations for an orbit with a given p_z and $\epsilon = \epsilon_0$: the error in an incomplete rotation is insignificant — it makes a non-resonant contribution to j_α.

Equations (2.1) and (2.5) can be continued as even functions in the region $y < 0$ if we set $E_\alpha(-y) = E_\alpha(y)$ and substitute $s(y - r(t) - r_0)$ for $s(|y - r(t)| - r_0)$, which is equal to it when $y > 0$; it is easy to see [if we take into account the central symmetry of the Fermi surface: $\epsilon(\mathbf{p}) = \epsilon(-\mathbf{p})$] that in this case $j_\alpha(y) = j_\alpha(-y)$.

The basic mathematical difference from the calculations of reference 1 is that in the latter case the minimum parameter is the "anomaly" parameter δ_0/r and in the taking of asymptotes one assumes $\omega\tau \ll r/\delta_0$; in the present case the minimum parameter is $1/\omega\tau$, and $\omega\tau \gg r/\delta_0$. However, in exactly the same fashion as was done earlier,[1] it can be shown that account of the boundary condition (which leads, as we have seen, to the substitution of unity for $s[y - r(t) - r_0]$) leads only to multiplication of the quantities of interest to us by a constant of the order of unity.

Physically this is quite clear: the important role, in any case, could be played even in the absence of a boundary only by trajectories on which the electron covers in a skin layer a path on the order of $\sqrt{r\delta}$, i.e., close to the maximum possible value (trajectories 1, 2, and 3, but not 4, in Fig. 4). Account of the boundary eliminates the trajectories of type 1, 3, i.e., it merely decreases somewhat the effective conductivity. This decrease is the more insignificant in that the quantities of interest to us (for example, the impedance) are determined by the cube root of the effective conductivity. Therefore, for simplicity of presentation only, we shall disregard collisions of an electron with the surface, and investigate Eqs. (2.1) and (2.5) with unity in place of $s[y - r(t) - r_0]$.

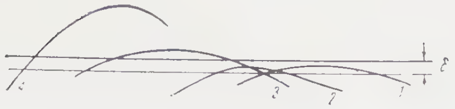

FIG. 4

Such a system of equations is quickly solved by the method of Fourier components, for which it suffices to multiply both sides of the equation by e^{iky} and integrate over y from $-\infty$ to $+\infty$. We have

$$-k^2\mathscr{E}_\alpha(k) - 2E'_\alpha(0) = iK_{\alpha\beta}(k)\mathscr{E}_\beta(k),$$

$$K_{\alpha\beta}(k) = \frac{8\pi\omega e^3 H}{c^3 h^3}\int\left[1 - \exp(-2\pi i\omega/\Omega - 2\pi/\Omega\tau)\right]^{-1} dp_z$$

$$\times\int_0^T \exp\{-i\omega t + ikr(t)\}v_\alpha(t)\,dt\int_0^T \exp\{i\omega t' - ikr(t')\}v_\beta(t')\,dt',$$

$$\mathscr{E}_\alpha(k) = \int_{-\infty}^{\infty} e^{iky}E_\alpha(y)\,dy,$$

$$E_\alpha(y) = \frac{1}{\pi}\int_0^\infty \mathscr{E}_\alpha(k)\cos ky\,dk, \tag{2.6}$$

whence it is easy to find $\mathscr{E}_\alpha(k)$ and $E_\alpha(y)$.

Although this solves the problem in principle, we carry out, for convenience of investigation of the function $E_\alpha(y)$, a further simplification of Eqs. (2.6). To determine the impedance and the "spikes" of current and field (the "what-not shelves" in Fig. 2), as is clear from Sec. 1, only electrons close to the "upper" and "lower" points of the central cross section are important, where $p_z = 0$, $v_y = 0$, so that $r(t)$ has a minimum or a maximum. This statement (as well as many others which will be encountered later, statements made without proof) can be rigorously verified.

At the points mentioned,

$$v_\beta(\varepsilon, p_z, p_x) \approx v_\beta(\varepsilon_0, 0, p_x^{max})\,\text{sign}\,v_x \equiv v_\beta^0\nu, \tag{2.7}$$

and $K_{\alpha\beta}$ [and with it Eq. (2.6)] is automatically diagonalized by the choice of the new orthogonal axes α and β, where α is the direction of the velocity v at the point $\epsilon = \epsilon_0$, $p_z = 0$, $p_x = p_x^{max}$, so that $v_\alpha^0 = v_0$. The fact that in this case $K_{\alpha\beta} = K_{\beta\alpha} = K_{\beta\beta} = 0$ apparently means only that $K_{\beta\beta}$, $K_{\alpha\beta}$, $K_{\beta\alpha}$ do not have a resonant character. Therefore, only E_α has "spikes" while E_β has none.

We note that we disregard cross terms in the determination of both E_α and E_β. [In the determination of E_β one must write down the non-resonant $K_{\beta\beta}$ correctly according to the formula (2.6).] This is connected with the fact that, as can be proved by direct calculation,

$$K_{\alpha\alpha} \sim (\omega\tau)^{-1/2}, \quad K_{\beta\beta} \sim K_{\alpha\beta} \sim K_{\beta\alpha} \sim 1,$$
$$E_\alpha \sim (\omega\tau)^{-1/4}, \quad E_\beta \sim 1.$$

Naturally, the diagonalized equation

$$-k^2\mathscr{E}_\alpha - 2E'_\alpha(0) = iK_{\alpha\alpha}\mathscr{E}_\alpha \tag{2.8}$$

is not suitable for the determination of E_α between the spikes, where all points of the orbit are important and not only its vertices. However, inasmuch as almost everywhere $v_{\alpha,\beta} \sim v_{\alpha,\beta}^0$ and the cross terms do not change the behavior of the functions, Eq. (2.8) is useful for a qualitative analysis of the behavior of E_α for all y, as can be established directly by simple (though cumbersome) calculations.

From (2.8) and (2.6), by setting

$$k = x/d_0, \quad d_0 = d|_{p_z} = 0; \qquad y = d_0\zeta \tag{2.9}$$

and omitting the index α everywhere in what follows, we get

$$E(\zeta) = -\frac{2d_0}{\pi}E'(0)\int_0^\infty \frac{\cos\zeta x\,dx}{x^2 + id_0^2K(x/d_0)}. \tag{2.10}$$

3. INVESTIGATION OF THE STRUCTURE OF THE FIELD IN A METAL

For simplicity, let us consider the case of "exact" resonance: $\omega = q\Omega_0$. It is easy to prove that for $\omega\tau \gg r/\delta_0$ one can set $p_z = 0$ in (2.6) in the inner integrals, so that we get from (2.10)

$$E(\zeta) = -\frac{2d_0}{\pi}E'(0)J(\zeta), \quad J(\zeta) = \int_0^\infty \frac{\cos\zeta x\,dx}{x^2 + i\exp(-i\pi\sigma/4)M^3A^2(x)},$$

$$A(x) = A(-x)$$
$$= \sqrt{|\dot{v}_y^0|/2\pi d_0}\left|\int_0^T \nu(t)\exp[-i\omega t + ixr(t,0)/d_0]\,dt\right|. \tag{3.1}$$

Here,

$$r = r(t, p_z), \quad M^3 = \frac{12\sqrt{2}\,qv_0^2\sqrt{\omega\tau_0}}{\Omega_0|\dot{v}_y^0|\sqrt{|m_0^{*\prime\prime}m_0^*|}}\frac{1}{\delta_0^2} \sim 10\,q^3\sqrt{\omega\tau_0}\left(\frac{r_1}{\delta_0}\right)^2,$$

$$1/\delta_0^2 = 2\pi e^2 8\pi(p_x^{max})^3(3c^2m_0^*h^3)^{-1}, \quad r_1 \sim v/\omega,$$
$$d_0/2 = r_{00} = p_x^{max}/m_0^*\Omega_0, \quad \sigma = \text{sign}(\Omega_0/\Omega'')_0. \tag{3.2}$$

Here the dot indicates the derivative with respect to t and the prime the derivative with respect to p_z; the index 0 denotes the point $\epsilon = \epsilon_0$, $p_z = 0$, $p_x = p_x^{max}$. [The fact that $A(-x) = A(x)$ can be established by making use of the substitution $t \to T_0/2 + t = \pi/\omega + t$.]

Taking it into account that

$$J = \int_0^\infty \ldots\cos\zeta x\,dx = \int_{-\infty}^{\infty}\ldots\exp(i\zeta x)\,dx, \tag{3.3}$$

where $A^2(x)$ is an analytic function, and applying the theorem of residues, we obtain for ζ not too close to an integer ($\zeta = a + \zeta'$; a is an integer, $\zeta' \gg M^{-1}$),

$$J \approx 2\pi M^{-3/2} e^{-i\pi(2-\sigma)/8} \sum_{n=1}^{\infty} \frac{\cos \zeta x_n \exp(-\zeta\beta_n)}{x_n \,|\, A'(x_n)\,|}; \qquad (3.4)$$

$$\beta_n = M^{-3/2} e^{-i\pi(2-\sigma)/8} \,|\, x_n / A'(x_n)\,|; \quad A(x_n) = 0, \quad x_n > 0. \qquad (3.4a)$$

We note that for $x \gg 1$

$$A(x) \sim \sqrt{2/x}\cos(x/2 - \pi/4), \qquad x_n \sim (2n + 1/2)\pi,$$

$$\beta_n \sim M^{-3/2} n^{3/2}. \qquad (3.5a)$$

If $\zeta' \sim 1$, the series (3.4) converges rapidly: only $n \sim 1$ are important. The function $J(\zeta)$ oscillates at distances of the order of unity $(y \sim r)$ and falls off extremely slowly at distances of the order of $M^{3/2}$ [$y \sim r^2(\omega\tau)^{1/4}/\delta_0$; for $\omega \sim 10^{11}\,\mathrm{sec}^{-1}$, $1 \sim 10^{-1}$ cm, $n \sim 10^{22}\,\mathrm{cm}^{-3}$, this corresponds to a depth of the order of 1 mm]; for $\zeta \gg M^{3/2}$ it is enough to keep only the first term in the series (3.4).

For $\zeta' \ll 1$, the terms $n \gg 1$ become important in the series (3.4), which makes it possible to make direct use of the asymptotic expression (3.5a) for $A(x)$ in the integral $J(\zeta)$. In this case, $J(\zeta)$ is conveniently calculated in the following fashion, with accuracy to M^{-3}. Transforming $J(\zeta)$:

$$J(\zeta) = \int_0^{\infty} \frac{x\cos(a + \zeta')\,x\,dx}{x^3 + iM^3\exp(-i\pi\sigma/4)(1 + \sin x)} = \sum_{n=0}^{\infty} \int_{2\pi n}^{2\pi(n+1)} dx \ldots$$

$$\approx \sum_{n=0}^{\infty} \int_0^{2\pi} \frac{2\pi n\cos(ax + 2\pi n\zeta')\,dx}{(2\pi n)^3 + i\exp(-i\pi\sigma/4)\,M^3(1 + \sin x)}$$

and replacing the sum by an integration, we obtain

$$J(\zeta) = \frac{1}{2\pi M}\int_0^{\infty} y\,dy \int_0^{2\pi} \frac{\cos(ax + \zeta' My)\,dx}{y^3 + i\exp(-i\pi\sigma/4)(1 + \sin x)}$$

$$= \frac{(-1)^a}{M}\int_0^{\infty} \frac{y\cos(\zeta' My + \pi a/2)\,dy}{\sqrt{g^2 - 1}\,(g + \sqrt{g^2 - 1})^a},$$

$$g = 1 + y^3\exp[-i\pi(2 - \sigma)/4],$$

$$\sqrt{g^2 - 1} \equiv g\sqrt{1 - 1/g^2}, \qquad \sqrt{1} = +1. \qquad (3.5)$$

Equations (3.5) are also suitable for the calculation of the derivatives of $J^{(k)}(\zeta)$ which are used below in Sec. 6 [in the series for derivatives, obviously, $n \gg 1$ are important for any ζ, but substitution of summation for integration, i.e., keeping only the zero term in the Poisson formula which can be readily justified for $\zeta' \ll 1$ and $k \geq 0$, and the transition to a formula analogous to (3.5), are no longer valid when $\zeta' \sim 1$.]

4. CONCLUSIONS FROM ANALYSIS OF THE BEHAVIOR OF THE FIELD

Let us formulate without proof the results of the investigation of Eqs. (2.10), (3.4), (3.5). These results substantiate the conclusions of Sec. 1 (see Fig. 2).

1. Near $y \approx d_0, 2d_0, \ldots$, the absolute value of the field has a sharp maximum and is $\sqrt{M}$ times greater than the average field between the maxima.

2. The distance from a maximum at which a value of the order of the maximum M^{-1} is achieved corresponds to a distance y from $\delta_0(r/\delta_0)^{1/6}$ to δ_0. This is easily understood if we note that for $\zeta' M \gg 1$ we obtain $J(\zeta') \sim |\zeta' M|^{-1/2}$.

3. With increase in the number of the maximum, its height falls off (for $a \gg 1$) as $a^{-1/3}$, and the "width" grows. The sharp maxima gradually disappear; at large distances the field oscillates over distances $y \sim r$ and slowly decays at a depth $rM^{3/2} \sim (r^2/\delta_0)(\omega\tau)^{1/4}$.

4. For $y_b = 2bd_0\,(b = 0, 1, 2, \ldots)$ there are separate extrema of the value of the field, the signs of which alternate as $(-1)^b$.

5. For $y \approx y_b = (2b + 1)\,d_0\,(b = 0, 1, 2, \ldots)$ there are two extrema which differ only in sign; the field close to these points is antisymmetric; $E(y_b + y') = -E(y_b - y')$. The signs of the first of the two neighboring extrema alternate as $(-1)^{b+1}$; the first extremum at $y \approx d_0$ has the sign opposite the sign of the field for $y = +0$. The current density changes in a fashion similar to the field intensity, with

$$j(\zeta') \sim |\,|\zeta' M| + 1|^{-1}, \qquad j(a) \sim a^{-1} \qquad (a \gg 1).$$

6. For $\Omega_0\tau \ll r/\delta_0$, the relative growth in the field close to $y = d_0$ is of the order $(\Omega_0\tau)^{1/2}$; the a-th spike is of order

$$E(0)\,a^{-1/3}(\delta_0/r)^{a/3}(\Omega_0\tau)^{5a/12}.$$

Thus, the condition $\Omega_0\tau \geq r/\delta_0$ is extremely important, for unless it is fulfilled the spikes decay rapidly. These conclusions can be obtained from (2.6) where naturally it is not possible to set $p_z = 0$ in the inner integrals, and the term periodic for $kr \gg \omega\tau$ has the order $\sqrt{\omega\tau/kr}$.] Here, inasmuch as the effective τ depends strongly on the degree of parallelism of the magnetic field to the surface of the metal, strict parallelism is essential if spikes are to be observed.

5. MATHEMATICAL REASON FOR THE PRESENCE OF SPIKES OF THE FIELD. CHARACTERISTIC FREQUENCIES OF PLASMA OSCILLATIONS

At first glance, the fact that the field rapidly decays in the interior appears to be an obvious consequence of Eq. (2.1). Actually, inasmuch as

$$j_\alpha(y) \sim (\sigma_{eff}/r)\int K_{\alpha\beta}(y, y')\,E_\beta(y')\,dy'$$

(σ_{eff} is the effective conductivity) and

$$\frac{1}{N} E''_\alpha \sim \frac{i}{r^2} \int \frac{1}{r} K_{\alpha\beta}(y, y') E_\beta(y')\, dy', \quad N \gg 1, \qquad (5.1)$$

while $K \sim 1$ and falls off at distances of the order r, then

$$E'' \sim iN\bar{E}\,/\,r^3, \quad \bar{E} \sim \int E(y')\, dy',$$

and E must fall off at distances of the order of $r/\sqrt[3]{N} \ll r$.

However, this conclusion is valid only when for the given $K_{\alpha\beta}(y, y')$ the integral

$$\int K_{\alpha\beta}(y, y') E_\beta(y')\, dy'$$

is of the order of $\bar{E}$, i.e., if (5.1) does not have a zero eigenvalue.

In the considered case, for an equation corresponding to (2.8), we have

$$(\omega\tau)^{-1/2} E''_\alpha = (\omega\tau)^{-1/2} i\hat{K}_{\alpha\alpha} E_\alpha = i\hat{K}^{(1)}_{\alpha\alpha} E_\alpha.$$

For $\omega = q\Omega_0$, where q is an integer, $\tau = \infty$ is an infinitely multiply degenerate eigenvalue with eigenfunctions $A_n \cos(x_n y/d_0)$, where x_n is a root of (3.4a). We are extremely close to this eigenvalue and it is not difficult to see that (3.4) is a superposition of eigenfunctions, and that $E(\xi)$, averaged over the interval $M^{-1} \ll \Delta\xi \ll 1$, is close to the eigenfunction. The "spikes" in this case are the natural singularities of the solution of the equation close to its eigenvalue.

It is interesting that as $\tau \to \infty$ the natural frequencies ω_q of the plasma oscillations in a strong magnetic field ($\Omega_0\tau \gg 1$) are shown in this fashion to be discrete multiples of Ω_0: $\omega_q = q\Omega_0$ (this means several orders of magnitude less than the usual natural frequencies $\omega_0 \sim c/\delta_0$) and the corresponding discrete values of the wave vector

$$k_n = x_n / d_0 = \omega_q x_n m^* / 2qp_x^{max}.$$

6. DETERMINATION OF THE SURFACE IMPEDANCE

We shall investigate how the phenomenon of "spikes" affects the surface impedance. We immediately note that, inasmuch as the impedance has a minimum at cyclotron resonance, a significant resonance will be observed, as noted earlier,[1] only for a definite polarization of the electromagnetic wave, when **E** is directed along $\mathbf{v}_0$. Therefore, as has been shown,[4] it is more convenient to measure the derivative of the impedance with respect to the magnetic field, which is determined essentially by $Z_{\alpha\alpha}$, where

$$Z_{\alpha\alpha} = (-4\pi i\omega / c^2)(E_\alpha(0)/E'_\alpha(0)) \equiv Z.$$

Making use of (3.1), and carrying out transformations similar to those applied in Sec. 3, we find

$$Z = \frac{4i\omega d_0}{\pi c^2 M} (q\sqrt{\varkappa})^{1/3} \int_0^\infty z\,dz$$

$$\cdot \int_0^{2\pi} \left[z^3 + \frac{\sigma}{\pi} \int_{-\infty}^{\infty} \frac{\{1 + \sin(x + s_2 N_1 z w^2)\}\, dw}{w^2 + s\sigma \exp(-is \tan^{-1}|\varkappa_1|^{-1})} \right]^{-1} dx; \qquad (6.1)$$

where

$$\varkappa = \sqrt{1 + \varkappa_1^2}, \qquad \varkappa_1 = (\omega - q\Omega_0)\tau_0,$$

$$s = \operatorname{sign}(\omega - q\Omega_0), \quad s_2 = \operatorname{sign} r''_0,$$

$$N_1 = |r''_0 m^*_0 / r_0 m^{*''}_0| M\varkappa^{5/6}(\omega\tau_0)^{-1} \sim (r/\delta_0)^{2/3} \varkappa^{5/6} (\omega\tau_0)^{-5/6}. \qquad (6.2)$$

The integration in (6.1) is easily carried out; the equations obtained which are too complicated to write down, make it possible to plot the entire resonance curve. Thus, for $s\sigma = 1$, $N_1 \ll 1$, we have

$$Z(N_1) \approx Z(0) \Big\{ 1 - 2^{2/15} \pi^{-1/2} \Gamma\left(\frac{1}{3}\right) \Gamma\left(\frac{7}{6}\right)$$

$$\times N_1^{1/10} e^{i\varphi} \int_0^\infty \frac{dy}{y^2} \left[1 - \frac{1}{\sqrt{1 + y^6}} \right] \Big\},$$

$$\varphi = \frac{\pi}{60} \left(1 - \sigma + \frac{\sigma}{\pi} \tan^{-1}|\varkappa_1|^{-1} \right),$$

and comparing $Z(0)$ with $Z(\infty)$, we find that the appearance of "spikes" of the field increases the impedance (that is, it acts "against" the resonance) by a factor of $\Gamma(1/6)[\sqrt[3]{2}\sqrt{\pi}\,\Gamma(2/3)]^{-1} \approx 1.84$, compared with the "ordinary" case examined in reference 1, without changing all the dependences. This corresponds to a decrease of $Z(\infty)$ with changing $\omega\tau_0$ to approximately 1/40.

We shall make clear how the presence of "spikes" in the case of films of thickness $a = D/d_0 \gg 1$ affects the impedance and its derivatives. According to the definition of impedance in (3.1),

$$Z = E(0) / \int_0^D j(y)\, dy \approx Z_\infty \{1 + J'(a) / J'(0)\},$$

$$J'(0) = -\frac{\pi}{2} \qquad (6.3)$$

where Z_∞ is the impedance of the half-space and the integral $J(a)$ is given by Eq. (3.1). Thus for the determination of $Z^{(k)}(a)$ it is sufficient to know $J^{(k+1)}(a)$. (Strictly speaking, one should also take into account the presence of the surface $y = D$, writing down the boundary condition corresponding to it, extending E and its equation in even fashion in the region $0 > y > -D$ and beyond periodically through all space with period 2D, and solving the equation by expansion in a Fourier series. However, for $a \gg 1$, in the zeroth approximation, use can be made of the solution for $E(\zeta)$ in the case of a half-space.)

Without carrying out the detailed derivations here, we only note that the jump in $J^{(k+1)}$ is of the order $M^k a^{-(2k+3)/3}$, where even $a - k$ gives a single jump, and odd $a - k$ gives two jumps very close together, equal in magnitude and opposite in sign. Physically, this is evident from (6.3) and Fig. 2: if a is varied, the current has alternately a single jump and two jumps, and these compensate for each other.

7. NEW RESONANCE EFFECTS

The basic predictable effect is the gradually damped and broadened "spikes" of field and current density in the metal at a depth that is a multiple of the orbit corresponding to the central cross section of the Fermi surface, in a direction normal to the surface of the metal, and the slow damping of the field in the body of the metal [at distances of the order $r^2(\omega\tau)^{1/4}/\delta_0$]. The peculiarities of the structure of the field were analyzed in Sec. 4; they are shown in Fig. 2.

Such a structure of the field leads to a number of new resonance effects.

1. It is simplest to observe experimentally the change in the magnetic field of the derivatives of the impedance $Z = R + iX$ (dR/dH, dX/dH, $d\ln X/dH$), since the impedance itself has, as has been pointed out, a resonant character only for a definite polarization of the electromagnetic field. However, one could observe with certainty only a non-monotonic change of the impedance with the appearance of "spikes" (which is connected with the fact that the approach of the magnetic field to resonance leads to resonance reduction of the impedance and then to increase of it as a result of the appearance of "spikes") which does not always take place. It is more convenient to measure the quantity

$$\Delta = (H - H_{\text{res}})\, d\ln X/dH$$

(H_{res} is the resonance value of H). Away from resonance it is constant without "spikes"; the presence of spikes leads to an additional "bump" since for $N_1 \gg 1$ and for $N_1 \ll 1$ (see Sec. 6) we will have $\Delta \approx -1/6$. Naturally such a criterion for the appearance of "spikes" is not conclusive.

2. It appears to us that the effects on layers of thickness $D \sim 10^{-3} - 10^{-1}$ cm are the simplest of all observed and also clearly account for the "what-not shelves" in the field structure (Fig. 2).

Upon increase in the frequency ω, one ought to observe jumps of the resonance values of the impedance and its derivatives (the latter, as was pointed out above, are much more convenient), which correspond to the fundamental ($\omega = \Omega_0$) when $D = ad_0 \equiv a2cp_x^{\max}/eH$ (a is an integer), and also an abrupt increase in the number of observed harmonics by one (from a to $a + 1$), so that the number of observed harmonics also makes it possible to determine a and $d_0 = D/a$. The reason for this is clear from Fig. 5 (we recall that the change in the number of "spikes" of the field in the depth of the metal changes the impedance).

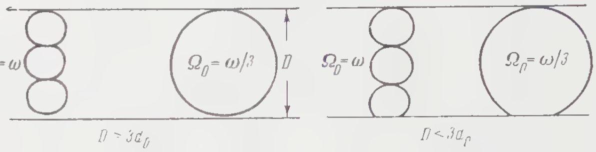

FIG. 5

A similar effect should also take place for fixed ω (which greatly simplifies the experiment) upon rotation of a constant magnetic field in the plane of the plate.

For angles φ at which the impedance changes abruptly and the last harmonic disappears, one can also determine the diameter of the Fermi surface $2p_x^{\max} = eHd_0/c$ in the corresponding directions. Naturally, in order to construct the Fermi surface directly, experiments are necessary on layers of different thickness and with different orientations of the surface relative to the crystallographic axes.

It is easy to estimate the magnitude and width in φ of the jumps in impedance and its derivatives due to rotation of the magnetic field. Since the jump corresponds to a change of ad_0 by a magnitude of the order of $a^{2/3}\delta_0$, then

$$\Delta\varphi \sim a^{2/3}\delta_0 / ad_0'(\varphi) \sim \delta_0 / a^{1/3} d_0,$$

that is, of the order of ten of seconds.

It is evidently most convenient to measure dZ/dH as a function of the angle of rotation φ and to study the variation of the tangent to the curve

$$f(\varphi) = H\,\frac{d(Z/Z_\infty - 1)}{dH} \approx a\,\frac{d(Z/Z_\infty - 1)}{da}.$$

Inasmuch as

$$\frac{df}{d\varphi} = \frac{\partial f}{\partial \varphi} + \frac{\partial f}{\partial a}\frac{da}{d\varphi}, \quad \frac{da}{d\varphi} \sim a,$$

it is easy, knowing J''' (see Sec. 6), to verify that the jumps of $df/d\varphi$ are of the order $M^2 a^{-1/3}$. Moreover, inasmuch as $M^2 \gtrsim (r/\delta_0)^{5/3} \sim 10^3$, one can hope to observe jumps in plates of thickness up to $M^{3/2} d_0 \sim 100 d_0$, that is, with $D \sim 1$ mm (because of the extremely slow damping of the "spikes," the thickness D is limited only by the inequality $D < M^{3/2} d_0$).

3. The third effect in which the "spikes" enter is the selective transmission of films at resonance, when the thickness of the film is $D = ad_0 \sim H$ (to observe this effect, it is obviously necessary to change the frequency ω and the magnetic field simultaneously or to rotate the magnetic field in the plane of the film), and to observe the field, which "penetrates" to a depth $r^2(\omega\tau)^{1/4}/\delta_0$. However, observation of these effects is very difficult because of the almost ideal reflectivity of the metal: reflection from only the two surfaces of the film leads to a reduction in the intensity of the transmitted wave by a factor $\sim (10\delta_0/\lambda)^2$, where λ is the wavelength of the incident wave. For a "penetrating" field which is not connected with the "spikes," the additional attenuation is of the order of $r/\delta_0 \sim v_0 \lambda (2\pi c \delta_0)^{-1}$.

4. The next effect is a "spatial" electron "echo," similar in a certain sense to the well-known spin "echo." If a pulse of duration $\Delta t \ll 2\pi/\Omega_0$ is incident on a metal placed in a constant magnetic field H such that $\Omega_0 \tau_0 \gg 1$, then within the time $2\pi/\Omega_0$, $4\pi/\Omega_0$, . . . (Ω_0 is the frequency corresponding to the central cross section), when the largest number of accelerating electrons reaches the surface of the metal, "echoes" will be observed — these are spikes of the field. (For a square dispersion law all of the accelerated electrons come together and the "echo" will be maximum.)

5. For $\Omega_0 \tau \gg 1$, the fluctuations in the metal will have an unusual character. This problem will be considered in a separate paper.

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SOVIET PHYSICS JETP VOLUME 12, NUMBER 2 FEBRUARY, 1961

ON THE GROWTH OF MAGNETOHYDRODYNAMIC WAVES IN A PLASMA STREAM MOVING THROUGH AN IONIZED GAS

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A dispersion equation is obtained which describes the propagation of electromagnetic waves in a plasma stream moving through an ionized gas. The low-frequency case (below the ion gyrofrequency), when the waves degenerate into magnetohydrodynamic waves, is considered in detail. It is shown that the system becomes unstable if the flow velocity exceeds the velocity of Alfven waves in the stream-plus-stationary-plasma system. In this case one of the normal waves builds up with time.

We consider the problem of magnetohydrodynamic waves in a plasma stream moving with a velocity $\mathbf{V}_0$ through a stationary ionized gas in the presence of a uniform external magnetic field H_0. The velocity of the stream, $\mathbf{V}_0$, is everywhere parallel to the lines of force of this field. The plasma in the infinite stream (number density N_s) and the plasma in the stationary medium (number density N_p) are assumed to be fully ionized and quasi-neutral.

The growth of high-frequency electromagnetic waves in ion and electron streams moving through plasmas have been studied in a number of works.[1-4] We investigate under what conditions the system (plasma stream plus stationary plasma) produces magnetohydrodynamic waves which increase in time and, consequently, when the system becomes unstable.

The linearized equations for the electrodynamic processes in this system are[5,6]

$$\frac{\partial \mathbf{j}_s}{\partial t} + (\mathbf{V}_0\nabla)\,\mathbf{j}_s + \nu_{ei}^s\,\mathbf{j}_s + \omega_H\,[\mathbf{j}_s\times\tau] = \frac{e^2 N_s}{m_e}\left(\mathbf{E} + \frac{1}{c}\,[\mathbf{V}_0\times\mathbf{h}] + \frac{1}{c}\,[\mathbf{v}_s\times\mathbf{H}_0]\right), \tag{1}$$

$$\frac{\partial \mathbf{v}_s}{\partial t} + (\mathbf{V}_0\,\nabla)\,\mathbf{v}_s = \frac{1}{\rho_s c}\,[\mathbf{j}_s\times\mathbf{H}_0], \tag{2}$$

$$\frac{\partial \mathbf{j}_p}{\partial t} + \nu_{ei}^p\,\mathbf{j}_p + \omega_H\,[\mathbf{j}_p\,\tau] = \frac{e^2 N_p}{m_e}\left(\mathbf{E} + \frac{1}{c}\,[\mathbf{v}_p\times\mathbf{H}_0]\right), \tag{3}$$

$$\frac{\partial \mathbf{v}_p}{\partial t} = \frac{1}{\rho_p c}\,[\mathbf{j}_p\times\mathbf{H}_0], \tag{4}$$

$$\operatorname{curl}\mathbf{E} = -\frac{1}{c}\,\frac{\partial \mathbf{h}}{\partial t}, \tag{5}$$

$$\nabla^2\mathbf{E} - \frac{1}{c^2}\,\frac{\partial^2\mathbf{E}}{\partial t^2} = \frac{4\pi}{c^2}\,\frac{\partial\,(\mathbf{j}_p + \mathbf{j}_s)}{\partial t}. \tag{6}$$

Here, $\mathbf{j}_s$, $\mathbf{v}_s$, and $\rho_s = (m_e + m_i)\,N_s$ are the electric current density, the velocity, and the mass density of the ionized gas in the stream; $\mathbf{j}_p$, $\mathbf{v}_p$ and ρ_p are the analogous quantities for the stationary plasma; ν_{ei} is the frequency of collisions of electrons with ions; $\omega_H = eH_0/m_e c$ is the gyrofrequency of the electron (m_e is the electron mass); $\mathbf{E}$ and $\mathbf{h}$ are the electric and magnetic field strengths; τ is a unit vector in the direction of the magnetic field $\mathbf{H}_0$; the subscripts s and p relate to quantities measured in the streaming and stationary plasma, respectively. Equations (2) and (4) are the hydrodynamic equations of a plasma and the equations for the current [(1) and (3)] express the generalized Ohm's law in the moving and stationary plasmas. Equation (6) is easy to derive from Maxwell's equations. In contrast to the equations given in reference 5 and 6, we have retained the total time derivative d/dt in the equations for the currents $\mathbf{j}_s$ and $\mathbf{j}_p$, as this is important in the case we are interested in.

Equations (1) to (6) lead readily to a dispersion relation for plane electromagnetic waves propagated along the flow and, consequently, along the lines of the magnetic field, $\mathbf{H}_0$. We use a cartesian coordinate system with the z axis directed along the magnetic field. We introduce the symbols $E_\pm = E_x \pm iE_y$, $j_{p\pm} = j_{px} \pm ij_{py}$ and similar symbols for j_s, v_s and v_p. By using these symbols and assuming that all the quantities vary as $\exp\{i(\omega t - kz)\}$, we obtain the following dispersion equation:

$$\omega^2 - c^2k^2 = \frac{\omega_{se}^2\,(\omega - kV_0)}{\omega - kV_0 \pm \omega_H - i\nu_{ei}^s - \omega_H\,\Omega_H\,/\,(\omega - kV_0)} + \frac{\omega_{pe}^2\;\omega}{\omega \pm \omega_H - i\nu_{ei}^p - \omega_H\Omega_H/\omega}, \tag{7}$$

where

$$\omega_{se} = (4\pi e^2 N_s / m_e)^{1/2},$$

$$\omega_{pe} = (4\pi e^2 N_p / m_e)^{1/2}, \qquad \Omega_H = eH_0 / m_i c.$$

The plus sign corresponds to the ordinary wave and the minus sign to the extraordinary wave. In the absence of a flow ($\omega_{se} = 0$), (7) becomes the well-known equation for electromagnetic waves propagated in a plasma along a magnetic field.[7] If the entire plasma moves, when $\omega_{pe} = 0$, we arrive, neglecting ions, at the equation derived by Twiss.[4]

We consider the usual approximation of magnetohydrodynamics — the case of very low frequencies ($\omega \ll \Omega_H$, $\omega/k \ll c$) — and neglect collisions for simplicity.[7] Furthermore, we assume that $kV_0 \ll \Omega_H$; Eq. (7) then takes on the much simpler forms

$$k^2 = (\omega - kV_0)^2 / V_{As}^2 + \omega^2 / V_{Ap}^2, \tag{8}$$

where we have introduced the Alfven velocity,

$$V_{As}^2 = \omega_H \Omega_H c^2 / \omega_{se}^2 = H_0^2 / 4\pi\rho_s, \quad V_{Ap}^2 = H_0^2 / 4\pi\rho_p.$$

The solution to Eq. (8) can be written in the form

$$\frac{\omega}{k} = \frac{V_0 \pm V_A \sqrt{(N_p / N_s)(1 - V_0^2 / V_A^2)}}{1 + N_p / N_s}, \tag{9}$$

where $V_A^2 = V_{As}^2 + V_{Ap}^2$. Thus, if the velocity of the stream exceeds the Alfven velocity ($V_0 > V_A$), waves appear in the stationary-plus-streaming plasma system. These waves build up in time, i.e., the system becomes unstable. This type of magnetohydrodynamic instability is different from the instabilities of plasma streams under electromagnetic perturbations of relatively high frequencies $\omega \geq \Omega_H$.[1-4] We observe that a similar criterion for instability occurs in magnetohydrodynamics when the instability of a tangential discontinuity is considered.[8]

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SOVIET PHYSICS JETP VOLUME 12, NUMBER 2 FEBRUARY, 1961

POLARIZATION OF PROTONS SCATTERED FROM O^{16}

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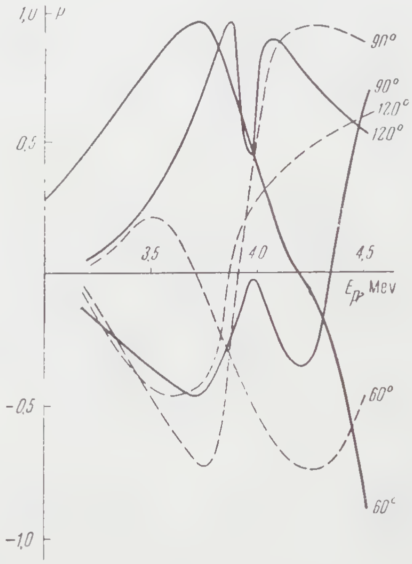
The polarization of protons elastically scattered from O^{16} has been calculated for protons in the energy range 3—4.5 Mev and for three scattering angles: 60°, 90° and 120°. It is shown that the polarization is very sensitive to the choice of level parameters for the F^{17} nucleus.

At the present time, only conflicting data are available in the literature on the energy levels of the F^{17} nucleus for excitation energies in the range 4 — 5.5 Mev.[1] According to the analysis of Laubenstein,[2] the experimental data given in reference[3] imply the following levels in this energy range: $3/2^+$ at 4.35 Mev and $3/2^+$ at 4.73 Mev. On the other hand, Sempert et al.[4] give different levels: $3/2^-$ at 4.5 Mev, $3/2^+$ at 4.6 Mev and $\frac{1}{2}^+$ at 5.1 Mev.

Measurements of the polarization in elastic scattering should resolve these difficulties. We have calculated the polarization P of protons elastically scattered from O^{16} for protons having energy in the range 3 — 4.5 Mev and for scattering angles 60°, 90° and 120° in the center-of-mass frame. The calculations were carried out within the framework of the resonance theory developed by Wigner and Eisenbud.[5] The results are shown on the figure. The solid curves are based on the energy levels given in reference,[2] while the dotted curves were calculated from the phases given in reference 4. It is evident that the polarization is quite sensitive to the choice of level parameters for the nucleus F^{17}.

The following points should be noted. In reference 2, the assumption was made that the S phase shift was large throughout a wide energy range. According to the Laubensteins, this might be connected with the existence of a wide, low lying $\frac{1}{2}^+$ level at 0.5 Mev. However, according to the data given by Sempert et al., the S-wave phase shift is no greater than would be expected from potential scattering. Our calculations indicate that this difference has very little effect on the magnitude of the polarization. The differences among the polarization curves are almost entirely due to the differences in assumed energies, spins and parities for the high-lying levels of F^{17}.



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SOVIET PHYSICS JETP VOLUME 12, NUMBER 2 FEBRUARY, 1961

PHOTOPRODUCTION OF π MESONS ON NUCLEONS IN PERIPHERAL COLLISIONS

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The production of π mesons in peripheral collisions between γ rays and nucleons is investigated. The photoproduction amplitudes are calculated using one-meson and two-meson approximations. The dispersion relations as well as the unitarity condition for the photoproduction amplitude and its analytical properties were used for this purpose. The deduced relations together with the corresponding experimental data make it possible to calculate the $(\gamma\pi\pi)$ vertex function near the point $t = 4\mu^2$, and the pion-nucleon coupling constant.

1. INTRODUCTION

THE theory of processes involving strongly interacting particles has recently met with considerable success by making use of the analytical properties and the unitarity of the scattering amplitudes. It was shown[1,2] that, for a number of processes, it is sufficient to know the properties of the scattering amplitudes in the region close to their nearest singularities in the transferred momentum. Okun' and Pomeranchuk[3] showed that the nearest singularities correspond to the largest impact parameters, and this fact served as a basis for a method of calculation of the amplitudes corresponding to large values of orbital angular momentum.

In the present article, this method is used for the study of peripheral collisions of photons with nucleons accompanied by meson production. The basis of the method is the dispersion relation for transferred momentum.[4] For the process in question, the relation is

$$A(s, t) = A_0(s) + \frac{a_1}{t-\mu^2} + \frac{1}{\pi}\int\limits_{4\mu^2}^{\infty}\frac{A_1(s, t')}{t'-t}dt'$$
$$+\frac{a_2}{u-m^2}+\frac{1}{\pi}\int\limits_{(m+\mu)^2}^{\infty}\frac{A_2(s, u')}{u'-u}du'. \quad (1)$$

where $A(s, t)$ is one of the invariant photoproduction amplitudes (in a general case, photoproduction is described by four amplitudes[5]), a_1 and a_2 are constants, and

$$-s=(p_1+k_1)^2=(p_2+k_2)^2, \quad -t=(p_2-p_1)^2=(k_1-k_2)^2,$$
$$u=2m^2+\mu^2-s-t,$$

where p_1 and p_2 are the initial and final four-momenta of the nucleon, k_1 is the momentum of the photon, and k_2 the momentum of the π meson. The coefficients of the amplitude expansion in terms of Legendre polynomials

$$a_l=\int\limits_{-1}^{1}A(s, t)P_l(\varkappa)d\varkappa$$

(where $\kappa = \cos\theta$, and θ is the angle between $\mathbf{p}_1$ and $\mathbf{p}_2$ in the c.m.s.) can be expressed in terms of Legendre polynomials of the second kind

$$Q_l(\varkappa_0)=\frac{1}{2}\int\limits_{-1}^{1}\frac{P_l(\varkappa)}{\varkappa_0-\varkappa}d\varkappa.$$

Using the relation $t = \mu^2 + 2\omega k(\kappa - 1/v)$, where μ is the π-meson mass, ω and k are the momenta in the c.m.s. of the photon and meson respectively, and $v = k/\sqrt{k^2+\mu^2}$ is the meson velocity, we obtain

$$a_l= -\frac{a_1}{\omega k}Q_l\left(\frac{1}{v}\right)+\frac{2}{\pi}\int\limits_{\bar{\varkappa}}^{\infty}A_1(s,t')Q_l(\varkappa')d\varkappa'$$
$$+\ldots, \quad \bar{\varkappa}=\frac{1}{v}+\frac{3\mu^2}{2\omega k}. \quad (2)$$

If, for a given l, we have $Q_l(\bar{\kappa}) \ll Q_l(1/v)$, then the main contribution is due to the first (polar) term in the scattering amplitude. The second term is important when $a_1 = 0$. The following two terms in Eq. (1) will not be considered since they correspond to still larger arguments of the function Q_l.

2. ONE-MESON APPROXIMATION

The polar term of the photoproduction amplitude can be obtained directly from the Feynman diagram in Fig. 1, corresponding to the exchange of one virtual meson. The vertices of this diagram are determined by the physical charge[6] of the meson, e, and the nucleon-nucleon interaction constant, g. We shall normalize the amplitude F_1 so that $d\sigma = |F_1|^2 do$ in the c.m.s. We then have

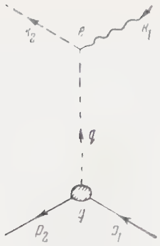

FIG. 1

$$F_1 = i\sqrt{2}\,eg\frac{m}{k\omega W}\sqrt{\frac{k}{\omega}}\,\frac{\mathbf{k}_2\chi}{1/v-\varkappa}\,\bar{u}_2\gamma_5 u_1, \tag{3}$$

where χ is the unit vector of the photon polarization, ω is the photon momentum, k is the meson momentum, $W = E_1 + \omega = E_2 + k_{20}$ is the total energy (all in c.m.s.), m is the nucleon mass, $e^2 = 1/137$, and $g^2 = 0.08\,(2m/\mu)^2$; u_i are four-component spinors normalized so that $\bar{u}_i u_i = 1$. Expressing these spinors in terms of two-component unit spinors v_i, we obtain $F_1 = v_2^* R_1 v_1$, where

$$R_1 = \frac{eg}{\sqrt{2}}\sqrt{\frac{k}{\omega}}\,\frac{\sqrt{(m+E_1)(m+E_2)}}{k\omega W}\,\frac{(\mathbf{k}_2\chi)}{1/v-\varkappa}$$
$$\times\left(\frac{\sigma\mathbf{k}_2}{m+E_2}-\frac{\sigma\mathbf{k}_1}{m+E_1}\right). \tag{4}$$

It should be noted that Eq. (3) does not exhibit the transversality property. However, the longitudinal part of Eq. (4) does not contain a pole.

In order to find the amplitude corresponding to a state of the nucleon-meson system with a given angular momentum j and parity $(-1)^{l+1}$, it is necessary to expand $R_1 v_1$ in terms of spherical spinors[7]

$$R_1 v_1 = \sum_{jlM} a^{(1)}_{jlM}\,\Omega_{jlM}(\mathbf{k}_2/k). \tag{5}$$

Using Eq. (4) and the explicit expressions for Ω_{jlM} we obtain

$$a^{(1)}_{jlM} = -\frac{eg\sqrt{\pi}}{\omega W}\sqrt{\frac{k}{\omega}}\sqrt{(m+E_1)(m+E_2)}$$
$$\times\left\{\frac{\omega}{m+E_1}\sqrt{\frac{l(l+1)}{2l+1}}\,(-1)^{1/2-\nu}C^{jM}_{l\lambda^1/_2\nu}\right.$$
$$\times\left[Q_{l-1}\left(\frac{1}{v}\right)-Q_{l+1}\left(\frac{1}{v}\right)\right]$$
$$\left.+\frac{k}{m+E_2}\sqrt{\frac{l'(l'+1)}{2l'+1}}C^{jM}_{l'\lambda^1/_2\nu}\left[Q_{l'-1}\left(\frac{1}{v}\right)-Q_{l'+1}\left(\frac{1}{v}\right)\right]\right\}, \tag{6}$$

where $l' = 2j - l$; $\lambda = \pm 1$ correspond to right- and left-handed polarizations of the photon; $\nu = \pm 1/2$; and $C^{jM}_{l\lambda^1/_2\nu}$ are the vector addition coefficients. The indices λ and ν can assume only one value for a given M: for $M = \pm 3/2$, $\lambda = \pm 1$, $\nu = \pm 1/2$; for $M = \pm 1/2$, $\lambda = \pm 1$, $\nu = \mp 1/2$. The values of the coefficients $a^{(1)}_{jlM}$ are given in Table I.

The polar term of the type discussed is contained only in the photoproduction amplitude of one type. Firstly, it is absent in the neutral π-meson production amplitude. Secondly, it corresponds to only one of the four possible types of the amplitude variation with the polarization of the nucleons and of the photon. [It can be seen from Eq. (3) that, e.g., F_1 vanishes if the meson moves in the direction of motion of the photon.] In order to find the other amplitudes, and also to estimate the accuracy of the one-meson approximation, it is necessary to analyze the integral term in Eq. (1). An analogous study in the case of nucleon-nucleon scattering was carried out by Galanin, Grashin, Ioffe, and Pomeranchuk.[8]

3. TWO-MESON APPROXIMATION

In order to study the integral term in Eq. (1), it is necessary to make use of the fact that, for $t > 4m^2$, $s < 0$, and $u < 0$, the quantity $A(s, t)$ represents the amplitude for the production of a

TABLE I. Values of the coefficients $a^{(1)}_{jlM}$ in units of $10^{-4}eg/\mu$

	ω, **Mev**	152.5	178.7	244.5
	$1/v$	2	1.5	1.2
$a^{(1)}_{l+1/2,\,l,\,-3/2} = -a^{(1)}_{l+1/2,\,l,\,3/2}$	$l=2$	137	282	479
	$l=3$	32.6	93.5	215
	$l=4$	8.14	32.8	104
	$l=5$	2.08	11.9	51.3
$-a^{(1)}_{l-1/2,\,l,\,-3/2} = -a^{(1)}_{l-1/2,\,l,\,3/2}$	$l=2$	298	607	1020
	$l=3$	56.7	161	365
	$l=4$	12.7	51.0	158
	$l=5$	3.06	17.3	73.8
$-a^{(1)}_{l-1/2,\,l,\,-1/2} = -a^{(1)}_{l-1/2,\,l,\,1/2}$	$l=2$	85.9	159	214
	$l=3$	24.0	63.1	120
	$l=4$	6.55	24.6	65.7
	$l=5$	1.78	9.54	35.8
$-a^{(1)}_{l+1/2,\,l,\,-1/2} = a^{(1)}_{l+1/2,\,l,\,1/2}$	$l=2$	88.6	163	220
	$l=3$	23.5	62.0	117
	$l=4$	6.27	23.6	62.7
	$l=5$	1.67	8.98	33.7

nucleon-antinucleon pair in a photon-meson collision: $\gamma + \pi \rightarrow N + \bar{N}$. In that range, $A_1(s, t) = \operatorname{Im} A(s, t)$, and, in order to find this, one can use the unitarity relation

$$2 \operatorname{Im} \langle N\bar{N} | \gamma\pi \rangle = \sum_n \langle n | N\bar{N} \rangle \langle n | \gamma\pi \rangle^*,$$

where $\langle b | a \rangle$ denote the elements of the scattering matrix $T = -i(S-1)$, which differ from the corresponding amplitudes by normalization factors. The function $A_1(s, t)$ calculated from this relation is analytic, and can therefore be continued in the region $t < 4m^2$. In the region $4\mu^2 < t < 9\mu^2$, the only state n is the two-meson state, and the unitarity condition is represented by the diagram in Fig. 2. In this diagram, the vertices represent the analytical continuation of the amplitudes of annihilation $N\bar{N} \rightarrow \pi\pi$ or of scattering $\pi N \rightarrow \pi N$, and of the amplitude of photoproduction of mesons on mesons $\gamma\pi \rightarrow \pi\pi$. Let us introduce the notation

$$\langle N\bar{N} | \gamma\pi \rangle = (2\pi)^4 \delta^{(4)} (p_1 + k_1 - p_2 - k_2)\, mB / 2\sqrt{E_1 E_2 \omega k_{20}},$$

$$\langle \pi\pi | N\bar{N} \rangle = (2\pi)^4 \delta^{(4)} (p_1 + f_1 - p_2 - f_2)\, mB^{(\pi)} / 2\sqrt{E_1 E_2 \mathcal{E}_1 \mathcal{E}_2}$$

$$\langle \pi\pi | \gamma\pi \rangle^* = (2\pi)^4 \delta^{(4)} (k_1 + f_2 - k_2 - f_1)\, B^{(\gamma)} / 4\sqrt{\mathcal{E}_1 \mathcal{E}_2 \omega k_{20}}.$$

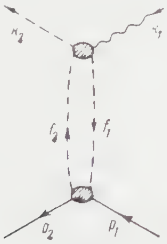

FIG. 2

where p_1 and p_2 are the nucleon four-momenta, E_1 and E_2 the nucleon energies, k_1 is the photon four-momentum, k_2 is the meson four-momentum, ω and k_{20} are the photon and meson energies respectively, f_1 and f_2 are the four-momenta of the intermediate mesons, and $\mathcal{E}_1$ and $\mathcal{E}_2$ their energies. The unitarity condition can now be written in the form

$$B_1 = \frac{1}{32\pi^2} \sqrt{\frac{t - 4\mu^2}{t}} \int B^{(\gamma)} B^{(\pi)} do,$$

$$B_1 = \operatorname{Im} B \text{ for } t > 4m^2,\ s < 0,\ u < 0, \qquad (7)$$

where do is the solid angle element in the c.m.s. corresponding to the annihilation channel $\mathbf{p}_1 = \mathbf{p}_2$, $\mathbf{f}_1 = \mathbf{f}_2$, $\mathbf{k}_1 = \mathbf{k}_2$. Introducing the invariants

$$z = \frac{\nu}{t - \mu^2} \sqrt{\frac{t}{t - 4m^2}}, \qquad z_1 = \frac{\nu_1}{\sqrt{(t - 4m^2)(t - 4\mu^2)}},$$

$$z_2 = \frac{\nu_2}{t - \mu^2} \sqrt{\frac{t}{t - 4\mu^2}}$$

(which correspond in the c.m.s. to the cosine of the angles between the vectors $\mathbf{p}_1, \mathbf{k}_1$; $\mathbf{p}_1, \mathbf{f}_1$; $\mathbf{f}_1, \mathbf{k}_1$) where, moreover,

$$\nu = PK, \qquad \nu_1 = Pf, \qquad \nu_2 = fK$$

$$(P = p_1 + p_2,\ K = k_1 + k_2, \qquad f = f_1 + f_2)$$

we can write Eq. (7) in the following form:

$$B_1 = \frac{1}{32\pi^2} \sqrt{\frac{t - 4\mu^2}{t}} \iint B^{(\gamma)} B^{(\pi)}$$

$$\times (1 - z^2 - z_1^2 - z_2^2 + 2zz_1z_2)^{-1/2} dz_1\, dz_2, \qquad (8)$$

where the integration is carried out over the area of the ellipse determined by the zero of the expression under the radical sign.

The structure of the amplitude $B^{(\gamma)}$ is uniquely determined by the condition that it should be a pseudoscalar quantity containing linearly the photon polarization vector χ and depending on the vectors K, f, and $q = k_1 - k_2$. The only such quantity is

$$B^{(\gamma)} = e_{\alpha\beta\lambda\mu}\, \chi_\alpha q_\beta K_\lambda f_\mu A^{(\gamma)}(\nu_2, t), \qquad (9)$$

where $A^{(\gamma)}$ is an invariant.

The amplitude $B^{(\pi)}$ has the following general form:

$$B^{(\pi)} = \bar{u}_2 \{ iA^{(1)}(\nu_1, t) + \hat{f} A^{(2)}(\nu_1, t) \} u_1. \qquad (10)$$

In Eq. (9) and (10) we have omitted the dependence on isotopic variables. This dependence is determined by the fact that two intermediate mesons should be found in the state with isospin T = 1. In fact, for T = 2, the meson could not have been emitted by nucleons; for T = 0, they are in an even-charge state and therefore could not have been emitted by a photon with its consecutive transformation into a π meson (which also has an even charge). For T = 1, all three mesons are in an odd-charge state, and the change in the isotopic state of the nucleon is determined by the charge of the emitted meson.

We thus have

$$B_1(\nu, t) = \frac{1}{32\pi^2} \sqrt{\frac{t - 4\mu^2}{t}}\, e_{\alpha\beta\lambda\mu} \chi_\alpha q_\beta K_\lambda \int A^{(\gamma)}(\nu_2, t) f_\mu$$

$$\times \bar{u}_2 [iA^{(1)}(\nu_1, t) + \hat{f} A^{(2)}(\nu_1, t)] u_1 do. \qquad (11)$$

The quantity B_1 is used for calculating an integral of the type of Eq. (2), in which the region near the lower limit $t = 4\mu^2$ makes the important contribution. Since, in Eq. (11), the interval of integration over ν_1 $(-1 \leq z_i \leq 1)$ is of the order of $\sqrt{t - 4\mu^2}$, we can use, instead of the functions which do not have singularities near this region, the first terms of their expansion at the point $\nu_i = 0$. From these considerations, we shall make the substitution

$$A^{(\gamma)}(\nu_2,\ t) \approx A^{(\gamma)}(0,\ t), \qquad A^{(1)}(\nu_1,\ t) \approx A^{(1)'}(0,\ t)\,\nu_1$$

($A^{(1)}(0, t) = 0$ because the function $A^{(1)}$ is odd with respect to ν_1.) Such an approximation is, however, not applicable for the function $A^{(2)}(\nu_1, t)$, since that function has no pole for $\nu_1 = \pm(t - 2\mu^2)$. Separating the polar term explicitly, let us write $A^{(2)}$ in the form

$$A^{(2)}(\nu_1,\ t) = A^{(p)}(\nu_1,\ t) + A^{(3)}(0,\ t),$$

$$A^{(p)}(\nu_1, t) = 4\pi g^2 \left(\frac{1}{t - 2\mu^2 - \nu_1} + \frac{1}{t - 2\mu^2 + \nu_1}\right). \qquad (12)$$

Thus

$$B_1 = \frac{1}{32\pi^2} \sqrt{\frac{t - 4\mu^2}{t}}\, e_{\alpha\beta\lambda\mu}\, \chi_\alpha q_\beta K_\lambda A^{(\gamma)}$$

$$\times (0,\ t)\bar{u}_2 \left\{ i A^{(1)'}(0,\ t) \int f_\mu \nu_1 do + A^{(3)}(0,\ t) \int f_\mu \hat{f}\, do + \int f_\mu \hat{f} A^{(p)}(\nu_1,\ t)\, do \right\} u_1. \qquad (13)$$

In view of the fact that the integrand in Eq. (13) does not depend on ν_2, we have

$$\int do \ldots = 2\pi \int_{-1}^{1} dz_1 \ldots .$$

and the integration can be carried out easily.

The amplitude $A^{(\gamma)}$ in Eq. (13) represents the physical amplitude for the photoproduction of mesons on mesons. Unfortunately, it is not possible to say anything definite about its value. An analysis of the meson photoproduction on nucleons and a comparison with the results of the present calculation may enable us to find its value. It should be noted that, in effect, our results will depend on the value of $A^{(\gamma)}$ at the one point $t = 4\mu^2$, $\nu = 0$.

The quantities $A^{(1)'}$ and $A^{(3)}$ represent the amplitudes for the scattering of mesons on nucleons in a non-physical region as a function of both variables t and ν_1. They were calculated in reference 8 by using the dispersion relations and experimental data. It should be noted that the procedure of analytical continuation of $A^{(1)}$ and $A^{(3)}$ from the physical region $t < 0$ used in reference 8 can be carried out only up to the region close to $t = 4\mu^2$. Our results will therefore be correct only to that extent in which the region close to the lower limit will be the only important one in integrals (2). The results obtained in reference 8 are as follows:

$$A^{(1)'}(0.4\,\mu^2) = -0.029 g^2/m\mu^2,$$

$$A^{(3)}(0.4\mu^2) = 0.025 g^2/2\mu^2.$$

These coefficients are small, and, since the integrals next to them in Eq. (13) are proportional to $(t - 4\mu^2)^{3/2}$ the contribution of these terms to B_1 will be small. We may, consequently, limit ourselves to the polar term in meson scattering $A^{(p)}$. Thus

$$B_1 = \frac{1}{4} g^2 \sqrt{\frac{t - 4\mu^2}{t}}\, e_{\alpha\beta\lambda\mu} \chi_\alpha q_\beta K_\lambda A^{(\gamma)}(0,\ t)$$

$$\times \bar{u}_2 \int_{-1}^{1} dz_1 f_\mu \hat{f} \left(\frac{1}{t - 2\mu^2 - \nu_1} + \frac{1}{t - 2\mu^2 + \nu_2}\right) u_1. \qquad (14)$$

Carrying out the integration in Eq. (14), we obtain the following expression for B_1

$$B_1 = \frac{1}{4} g^2 \frac{A^{(\gamma)}(0.4\mu^2)}{2i\mu m\,[(1 + x)(1 - \varepsilon^2 - \varepsilon^2 x)]^{1/2}}\, e_{\alpha\beta\lambda\mu} \chi_\alpha q_\beta K_\lambda \bar{u}_2$$

$$\times (c_1 \gamma_\mu + c_2 P_\mu \hat{P})\, u_1, \qquad (15)$$

where

$$c_1 = 4i\mu^2 \left\{ - \frac{\varepsilon(1 + 2x)\,[(1 - \varepsilon^2 - \varepsilon^2 x)\, x]^{1/2}}{2(1 - \varepsilon^2 - \varepsilon^2 x)} \right.$$

$$+ x \tan^{-1} \frac{2\,[x(1 - \varepsilon^2 - \varepsilon^2 x)]^{1/2}}{\varepsilon(1 + 2x)}$$

$$\left. + \frac{\varepsilon^2}{4} \frac{(1 + 2x)^2}{(1 - \varepsilon^2 - \varepsilon^2 x)} \tan^{-1} \frac{2\,[x(1 - \varepsilon^2 - \varepsilon^2 x)]^{1/2}}{\varepsilon(1 + 2x)} \right\},$$

$$c_2 = \frac{i\varepsilon^2}{1 - \varepsilon^2 - \varepsilon^2 x} \left\{ - \frac{3\varepsilon(1 + 2x)\,[x(1 - \varepsilon^2 - \varepsilon^2 x)]^{1/2}}{2(1 - \varepsilon^2 - \varepsilon^2 x)} \right.$$

$$+ x \tan^{-1} \frac{2\,[x(1 - \varepsilon^2 - \varepsilon^2 x)]^{1/2}}{\varepsilon(1 + 2x)}$$

$$\left. + \frac{3\varepsilon^2}{4} \frac{(1 + 2x)^2}{1 - \varepsilon^2 - \varepsilon^2 x} \tan^{-1} \frac{2\,[x(1 - \varepsilon^2 - \varepsilon^2 x)]^{1/2}}{\varepsilon(1 + 2x)} \right\}.$$

Let us denote by F_2 the amplitude of the process in the two-meson approximation, which is determined in the c.m.s. from the relation $d\sigma = |F_2|^2 do$. The discontinuity in the amplitude ΔF_2 near the point $t = 4\mu^2$, arising when t traverses a discontinuity along the real axis from the point $t = 4\mu^2$ to infinity, can, using Eq. (15), be written in the following form (δ is the isotopic index):

$$\Delta F_2^\delta = - \frac{g^2}{4\pi} \frac{m A^{(\gamma)}(0{,}4\mu^2)}{W} \sqrt{\frac{k}{\omega}}\, \varepsilon T_\delta \bar{u}_2 \left\{ \frac{W}{m} (\mathbf{k}_1 [\chi \times \mathbf{k}_2])\, J_2 \right.$$

$$\left. - [\omega \gamma [\mathbf{k}_2 \times \chi] + \gamma_0 \mathbf{k}_1 [\chi \times \mathbf{k}_2] + k_{20} \gamma [\chi \times \mathbf{k}_1]]\, J_1 \right\} u_1, \qquad (16)$$

where

$$J_1 = - \frac{\varepsilon}{2} \sqrt{x} + x \tan^{-1} \frac{2\sqrt{x}}{\varepsilon} + \frac{\varepsilon^2}{4} \tan^{-1} \frac{2\sqrt{x}}{\varepsilon},$$

$$J_2 = - \frac{3\varepsilon}{2} \sqrt{x} + x \tan^{-1} \frac{2\sqrt{x}}{\varepsilon} + \frac{3\varepsilon^2}{4} \tan^{-1} \frac{2\sqrt{x}}{\varepsilon}.$$

Expressing u_1 and u_2 through two-component spinors v_1 and v_2, we obtain

$$\Delta F_2^\delta = - \frac{g^2}{4\pi} \frac{m A^{(\gamma)}(0.4\mu^2)\, \varepsilon}{W} \sqrt{\frac{k}{\omega}}\, T_\delta \frac{\sqrt{(m + E_1)(m + E_2)}}{2m}$$

$$\times v_2^* \left\{ - \frac{W}{m} (\mathbf{k}_1 [\mathbf{k}_2 \times \chi])\, J_2 \left[1 - \frac{(\mathbf{k}_1 \mathbf{k}_2)}{(m + E_1)(m + E_2)} \right. \right.$$

$$- i \frac{\boldsymbol{\sigma}[\mathbf{k}_2 \times \mathbf{k}_1]}{(m + E_1)(m + E_2)} \Big]$$

$$+ J_1 \Big[ib\boldsymbol{\sigma\chi} + i \frac{\mathbf{k}_2\boldsymbol{\chi}}{m + E_2} \boldsymbol{\sigma}(\omega\mathbf{k}_2 - k_{20}\mathbf{k}_1) + \Big(1 + \frac{\omega}{m + E_1}$$

$$+ \frac{k_{20}}{m + E_2} + \frac{\mathbf{k}_2\mathbf{k}_1}{(m + E_1)(m + E_2)}\Big) \mathbf{k}_1 [\mathbf{k}_2 \times \boldsymbol{\chi}]$$

$$+ i \frac{\mathbf{k}_1 [\mathbf{k}_2 \times \boldsymbol{\chi}]}{(m + E_1)(m + E_2)} \boldsymbol{\sigma}[\mathbf{k}_2 \times \mathbf{k}_1]\Big]\Big\} v_1,$$

$$b = (\omega\mathbf{k}_2\mathbf{k}_1 - k_{20}\omega^2) / (m + E_1) + (k_{20}\mathbf{k}_2\mathbf{k}_1 - \omega k^2) / (m + E_2). \tag{17}$$

Let us write F_2 in the form $F_2 = v_2^* R_2 v_1$. The amplitude corresponding to the state with angular momentum j and parity $(-1)^{l+1}$ is given by the expression

$$a_{jlM}^{(2)} = \int \Omega_{jlM}^* \left(\frac{\mathbf{k}_2}{k}\right) R_2 v_1 do, \tag{18}$$

where the integration is carried out over the angles determining the direction $\mathbf{k}_2$. Assuming that $l \gg 1$ and $l\xi \gg 1$ ($\xi = \mu/\omega < 1$), we bring the integral over $d\kappa$ to the form

$$\int_{-1}^{1} P_l(\varkappa) R_2 d\varkappa = \frac{2\mu^2}{\pi i\omega k} Q_l(\varkappa_2) \int_0^\infty \exp(-L_l x) \Delta R_2 dx. \tag{19}$$

where κ_2 corresponds to the point $t = 4\mu^2$, and

$$L_l = (l + 1) F = 4\xi (l + 1) [16 + 9\xi^2$$
$$+ 6\xi^2 / (1 + \sqrt{1 + \xi^2/\varepsilon^2})]^{-1/2}.$$

A small region of integration near the lower limit is important in Eq. (19). It is therefore necessary to know the amplitude in the vicinity of the point $t = 4\mu^2$. The size of the effective integration interval is $\kappa \sim 1/L_l$ or $t - 4\mu^2 \sim 4\mu^2/L_l \ll 4\mu^2$. Let us substitute Eq. (19) into Eq. (18) and integrate over $d\varphi$ using Eq. (17). In the expression thus obtained, there remain integrals of the type

$$\int_0^\infty \exp(-L_l x) J_i dx.$$

Substituting the explicit expression for J_1, J_2, we obtain

$$\int_0^\infty \exp(-L_l x) J_1 dx =$$
$$- \frac{\pi\varepsilon}{8L_l} \left(\frac{2}{\sqrt{\pi L_l}} - \frac{\varepsilon}{2} + \frac{2\varepsilon}{3\sqrt{\pi}} \zeta_l \right),$$

$$\int_0^\infty \exp(-L_l x) J_2 dx =$$
$$- \frac{3\pi\varepsilon}{8L_l} \left(\frac{4}{3\sqrt{\pi L_l}} - \frac{\varepsilon}{2} + \frac{8\varepsilon}{9\sqrt{\pi}} \zeta_l \right). \tag{20}$$

The final result is expanded in terms of the parameter $\zeta_l = \epsilon\sqrt{L_l}/2$, which means that we consider $l \ll 4/\epsilon^2 \approx 180$.

Thus, the final expression for $a_{jlM}^{(2)}$ is

$$a_{l+1/2,\, l,\, -3/2}^{(2)} = - a_{l+1/2,\, l,\, 3/2}^{(2)} = A\beta(1/v)\,\Phi_1(1/v),$$
$$a_{l-1/2,\, l,\, -3/2}^{(2)} = a_{l-1/2,\, l,\, 3/2}^{(2)} = A\beta(1/v)\,\Phi_2(1/v),$$
$$a_{l+1/2,\, l,\, -1/2}^{(2)} = - a_{l+1/2,\, l,\, 1/2}^{(2)} = A\beta(1/v)\,\Phi_3(1/v),$$
$$a_{l-1/2,\, l,\, -1/2}^{(2)} = a_{l-1/2,\, l,\, 1/2}^{(2)} = - A\beta(1/v)\,\Phi_4(1/v). \tag{21}$$

Where the following notation is used:

$$A = \frac{\mu^2\varepsilon g^2 A^{(\gamma)}(0, 4\mu^2)}{4\sqrt{2}\,\pi} (T_\delta^{(+)} A^{(+)} + T_\delta^{(-)} A^{(-)} + T_\delta^{(0)} A^{(0)}),$$

$$\beta = (\varepsilon/4FW)[\pi k (m + E_1)(m + E_2)/\omega]^{1/2},$$

$$\Phi_1 = \sqrt{l(l + 1)(l + 2)} \Big\{ \frac{3W}{m} \frac{1}{2l + 1} (S_{l-1} - S_{l+1}) - (1 + h_1)$$
$$\times \frac{1}{2l + 1} (N_{l-1} - N_{l+1}) - \frac{3W}{m} h_1 h_2 \frac{1}{2l + 3} (S_l - S_{l+2})$$
$$- h_2 (1 + h_1) \frac{1}{2l + 3} (N_l - N_{l+2}) \Big\}, \tag{22}$$

$$\Phi_2 = \sqrt{(l - 1) l (l + 1)} \Big\{ \frac{3W}{m} \frac{1}{2l + 1} (S_{l-1} - S_{l+1}) - (1 + h_1)$$
$$\times \frac{1}{2l + 1} (N_{l-1} - N_{l+1}) - \frac{3W h_2}{m} \frac{h_1}{2l - 1} (S_{l-2} - S_l)$$
$$- h_2 (1 + h_1) \frac{1}{2l - 1} (N_{l-2} - N_l) \Big\}, \tag{23}$$

$$\Phi_3 = \sqrt{l + 1} \Big\{ \frac{3W}{m} \frac{l}{2l + 1} (S_{l-1} - S_{l+1}) - (1 - h_1) \frac{l}{2l + 1}$$
$$\times (N_{l-1} - N_{l+1}) + 2 \left(h_1 + \frac{k_{20}}{m + E_2}\right) N_{l+1}$$
$$- \frac{3W}{m} h_1 h_2 \frac{l + 2}{2l + 3} (S_l - S_{l+2})$$
$$+ h_2 (1 - h_1) \frac{l + 2}{2l + 3} (N_l - N_{l+2}) - \frac{2}{k} (k_{20} h_1 + k h_2) N_l \Big\}, \tag{24}$$

$$\Phi_4 = \sqrt{l} \Big\{ \frac{3W}{m} \frac{l + 1}{2l + 1} (S_{l-1} - S_{l+1}) - (1 - h_1) \frac{l + 1}{2l + 1}$$
$$\times (N_{l-1} - N_{l+1}) - 2 \left(h_1 + \frac{k_{20}}{m + E_2}\right) N_{l-1}$$
$$- \frac{3W}{m} h_1 h_2 \frac{l - 1}{2l - 1} (S_{l-2} - S_l)$$
$$+ h_2 (1 - h_1) \frac{l - 1}{2l - 1} (N_{l-2} - N_l) + \frac{2}{k} (k_{20} h_1 + k h_2) N_l \Big\},$$

$$S_l = \frac{1}{l + 1} Q_l(\varkappa_2) \left(\frac{4}{3\sqrt{\pi L_l}} - \frac{\varepsilon}{2} + \frac{8\varepsilon}{9\sqrt{\pi}} \zeta_l \right),$$

$$N_l = \frac{1}{l + 1} Q_l(\varkappa_2) \left(\frac{2}{\sqrt{\pi L_l}} - \frac{\varepsilon}{2} + \frac{2\varepsilon}{3\sqrt{\pi}} \zeta_l \right),$$

$$h_1 = \omega/(m + E_1), \quad h_2 = k (m + E_2). \tag{25}$$

The values of $T_\delta^{(+)}$, $T_\delta^{(-)}$, and $T_\delta^{(0)}$ are shown in Table III, which is taken from reference 5. The values of $\beta(1/v)\,\Phi_n(1/v)$ ($n = 1, 2, 3, 4$) are presented in Table II.

CONCLUSION

A rough estimate of the two-meson amplitude can be made if we assume the cross section for the photoproduction of mesons on mesons to be

TABLE II.

	ω, Mev	152.5	178.7	244.5		ω, Mev	152.2	178.7	244.5
	$1/v$	2	1,5	1,2		$1/v$	2	1,5	1,2
$\beta\,\Phi_1 \cdot 10^{-6}$	$l=3$	53.7	298	2850	$\beta\,\Phi_3 \cdot 10^{-6}$	$l=3$	45.5	254	2380
	$l=4$	3.71	32.9	525		$l=4$	3.31	29.3	463
	$l=5$	0.289	4.08	109		$l=5$	0.265	3.75	99.0
$\beta\,\Phi_2 \cdot 10^{-7}$	$l=3$	—105	—533	—4470	$\beta\,\Phi_4 \cdot 10^{-6}$	$l=3$	46.2	267	2210
	$l=4$	1.42	18.8	362		$l=4$	3.38	28.2	398
	$l=5$	0.447	7.19	206		$l=5$	0.258	3.45	81.8

TABLE III.

	$\gamma+p\to\pi^0+p$	$\gamma+n\to\pi^0+n$	$\gamma+p\to\pi^++n$	$\gamma+n\to\pi^-+p$
$T^{(+)}$	1	1	0	0
$T^{(-)}$	0	0	$\sqrt{2}$	$-\sqrt{2}$
$T^{(0)}$	1	—1	$\sqrt{2}$	$\sqrt{2}$

$\sim e^2/\mu^2$. In that case, $A^{(\gamma)}(0.4\mu^2) \sim 4\pi e/\mu^3$, and the two-meson amplitude already makes a negligible contribution for $l \gtrsim 3$ as compared with the one-meson amplitude.

Eq. (21) contains three constants $A^{(+)}$, $A^{(-)}$, and $A^{(0)}$. In fact, the photoproduction process is characterized by a single constant in each particular case. For the photoproduction of π^0 mesons on protons, $\gamma p \to \pi^0 p$, this constant is $A^{(+)} + A^{(0)}$; for the $\gamma n \to \pi^0 n$ process it is $A^{(+)} - A^{(0)}$; and for the processes $\gamma p \to \pi^+ n$ and $\gamma n \to \pi^- p$ the constant is $\sqrt{2}\,(A^{(0)} + A^{(-)})$ and $\sqrt{2}\,(A^{(0)} - A^{(-)})$ respectively. $A^{(+)}$, $A^{(-)}$, and $A^{(0)}$ can be determined from the study of several different processes.

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DEPENDENCE OF THE INELASTIC COLLISION CROSS SECTIONS OF ATOMS AND IONS ON VELOCITY IN THE CASE OF PSEUDO-INTERSECTION OF THE LEVELS

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The velocity dependence of the inelastic collision cross sections for atoms and ions is examined in the case of pseudo-intersection of the levels of the system of colliding particles. The time dependence of the electron wave functions is taken into account in terms of the radius vectors of the nuclei. The perturbation matrix element in the Landau-Zener formula includes the usual stationary separation of the levels and also a term that takes into account the indicated dependence of the electron wave functions on the time. Under certain conditions the cross section vs. velocity curves may have two peaks.

WE assume that the nuclei of the colliding atoms move in a straight line and uniformly in the sense of classical mechanics. Then the probability of excitation of the atoms is determined by the impact parameter of the collision, p, and by the velocity of the relative motion of the nuclei, $\mathbf{R} = \mathbf{v}$.

If $v \ll e^2/\hbar$, it is natural to represent the wave function Ψ of the atomic electrons in the form of an expansion in the molecular wave functions χ_n with the nuclei stationary. The coordinates of the nuclei, $\mathbf{R}_a$ and $\mathbf{R}_b$, are contained as parameters in the Hamiltonian and in χ_n. If $R = |\mathbf{R}_a - \mathbf{R}_b| \to \infty$, the functions χ_n are transformed into products of the atomic wave functions corresponding to the given atomic state with index n (the products differ in the transposition of the electrons between the atoms). The eigenvalues E_n of the electron energy are also functions of R. The curves $E_n(R)$ and $E_{n'}(R)$ can, in principle, intersect. Most frequently, however, we seemingly encounter an intersection of these curves at the point $R = R_0$ where $E_n - E_{n'}$ so to speak vanishes and reverses sign on going through $R = R_0$. In fact, the curves $E_n(R)$ and $E_{n'}(R)$ pass very close to each other and then diverge again in such a way that $E_n - E_{n'}$ does not reverse its sign as R goes through R_0. Such a behavior of $E_n(R)$ is called a pseudo-intersection of the levels.

If there is no intersection or pseudo-intersection of $E_n(R)$ and $E_{n'}(R)$, then the probability of the (n, n′) transition is exponentially small at small v. It is proportional to $\exp(-v_0/v)$; the greater $|E_n - E_{n'}|$, the greater v_0. Therefore, in the case of slow collisions, the (n, n′) transitions are of practical importance only near the intersection or pseudo-intersection points of the levels E_n and $E_{n'}$. The theory of such transitions, given in the papers of Landau[1] and Zener,[2] seems to us insufficiently complete.* We shall show that the intersection of such transitions, expressed as a function of the velocity v, has in general two maxima.

In the case of slow inelastic collisions of atoms or ions, we can consider in the pseudo-intersection of the levels only the transition between the two terms near the "intersection" point $(R = R_0)$.

The complete function of the system, $\Psi(\mathbf{R}_a, \mathbf{R}_b, \ldots \mathbf{r}_i \ldots)$, satisfies the equation

$$i\hbar\partial\Psi/\partial t = H\Psi. \tag{1}$$

Let us expand this function in a series

$$\Psi = C_1\chi_1^0 + C_2\chi_2^0 + \sum_{n=3}^{\infty} C_n\chi_n, \tag{2}$$

where χ_1^0 and χ_2^0 are orthonormal to each other and to functions χ_n when $n \geq 3$. They are expressed linearly (see below) in terms of χ_1 and χ_2, which satisfy, for fixed nuclear coordinates $\mathbf{R}_a$ and $\mathbf{R}_b$ (n = 1, 2 ...), the following equation:

$$H(\mathbf{R}_a, \mathbf{R}_b, \ldots, \mathbf{r}_i \ldots)\chi_n(\mathbf{R}_a, \mathbf{R}_b, \ldots, \mathbf{r}_i \ldots) = E_n(R)\chi_n(\mathbf{R}_a, \mathbf{R}_b, \ldots \mathbf{r}_i \ldots), \tag{3}$$

where $R = |\mathbf{R}_a - \mathbf{R}_b|$ and $\mathbf{r}_i$ are the radius vectors of the electrons.

For the coefficients C_m we obtain

$$i\hbar\dot{C}_m = E_m^0 C_m + \sum_{n \neq m}\left[H_{mn} - i\hbar\left(\frac{\partial}{\partial t}\right)_{mn}^0\right]C_n, \tag{4}$$

where $(\partial/\partial t)_{mn}^0 = (\partial/\partial t)_{mn}$, $E_m^0 = E_m$, and $H_{mn} = 0$ if $m > n$ and $n > 2$. If $m \leq 2$ and

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*This theory is briefly developed in the book by Landau and Lifshitz.[3]

$n \leq 2$, then $H_{11} = E_1^0$, $H_{22} = E_2^0$, and $H_{12} = H_{21} = V$. Integration is carried out in the matrix elements only over the electron coordinates.

If we put

$$\chi_1^0 = q\chi_1 - \sqrt{1-q^2}\chi_2, \qquad \chi_2^0 = \sqrt{1-q^2}\,\chi_1 + q\chi_2 \tag{5}$$

and introduce a monotonic function $q(R)$ such that $q^2(R_0) = 1/2$, $q^2 \to 1$ when $R \gg R_0$ and $q^2 \to 0$ when $R \ll R_0$, then this function will satisfy the relations

$$q\sqrt{1-q^2} = V/(E_1 - E_2), \tag{6}$$

$$E_{12}^0 = (E_1 + E_2)/2 \pm (2q^2 - 1)(E_1 - E_2)/2. \tag{7}$$

If q tends to unity or to zero, then E_{12}^0 tends to E_{12} or to E_{21}, while χ_{12}^0 tends to χ_{12} or χ_{21}.

In the case of a pseudo-intersection of levels, the difference $E_1 - E_2$ $(E_1 > E_2)$ reaches a minimum at a certain distance $R = R_0$ between the nuclei of the colliding particles. Then, putting $2V = (E_1 - E_2)_{\min} \approx \text{const}$, we obtain the intersection of the levels E_1^0 and E_2^0. It is obvious that if $(E_1 - E_2)_{\min}$ is small, the perturbation of V will also be small.

Furthermore, neglecting in (4) the transitions of the system to the states with $n > 2$, we arrive at a system of equations

$$i\hbar\dot{C}_1 = E_1^0 C_1 + (V + K_{12}^0)C_2,$$

$$i\hbar\dot{C}_2 = (V + K_{12}^0)^* C_1 + E_2^0 C_2. \tag{8}$$

This system differs from the analogous Zener equations[2] in the presence of the quantity K_{12}^0. This term has the form

$$K_{12}^0 = -i\hbar\int \chi_1^0 \frac{\partial\chi_1}{\partial t}\,d\tau = -i\hbar\int \chi_1^0(\dot{\mathbf{R}}\ \nabla_a + \dot{\mathbf{R}}_b\nabla_b)\,\chi_2^0 d\tau$$

$$= -i\hbar(\dot{\mathbf{R}}_a\nabla_a + \dot{\mathbf{R}}_b\nabla_b)_{12}^0. \tag{9}$$

It takes into account the dependence of the electron wave functions χ^0 on the time through the radius vectors of the nuclei, which act as parameters.

The process is thus adiabatic and the matrix element K_{12}^0 can be considered constant in the vicinity of the point of "intersection." Actually,

$$K_{12} - K_{12}^0 = \frac{-i\hbar\dot{R}}{\sqrt{1-q^2}}\frac{dq}{dR} = -\frac{i\hbar V F_{12}^0 \dot{R}}{(E_1 - E_2)^2}, \tag{9a}$$

Since

$$\left(\frac{\partial}{\partial R}\right)_{12} - \left(\frac{\partial}{\partial R}\right)_{12}^0 = \frac{1}{\sqrt{1-q^2}}\frac{dq}{dR} = \frac{VF_{12}^0}{(E_1 - E_2)^2},$$

$$F_{12}^0 = \frac{d}{dR}(E_1^0 - E_2^0).$$

The difference $K_{12} - K_{12}^0$ is a function that assumes a large value at the point of pseudo-intersection and diminishes rapidly with increasing distance from this point. The difference $E_1 - E_2$ changes rapidly near $R = R_0$. This is due to the singularity of the quantities χ_1 and χ_2, since the Hamiltonian H itself does not have any singularity as a function of R. It is therefore natural to assume that $(\partial/\partial R)_{12}^0$ is a smooth function, since E_1^0 and E_2^0 are smooth functions, whereas $(\partial/\partial R)_{12}$ includes a rapidly varying term $VF_{12}^0/(E_1 - E_2)^2$ [if we neglect $(\partial/\partial R)_{12}$, then in general there will be no such transitions].

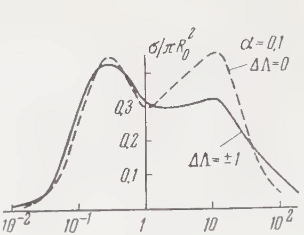

FIG. 1

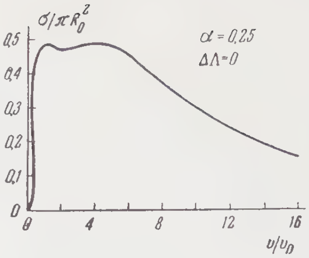

FIG. 2

We put

$$V' = V + K_{12}^0. \tag{10}$$

The solution of the system (8) can be found in the papers by Landau and Zener.[1,2] For the probability of transition of the system from state 2 into state 1 we obtain the Landau-Zener formula

$$w = 2e^{-S}(1 - e^{-S}), \tag{11}$$

$$S = 2\pi|V'|^2\ \hbar v_R|F_{12}^0|. \tag{12}$$

Here $v_R = \dot{R}$ is the radial component of the relative velocity of the particle, while $F_{12}^0 = d(E_1^0 - E_2^0)/dR$. Formula (11) was derived under more general assumptions, with allowance for the dependence of the electron wave functions χ_n^0 on the time through the parameters $\mathbf{R}_a$ and $\mathbf{R}_b$.

We note that the matrix element K_{12} has, generally speaking, Galilean invariance. Actually,

$$\dot{\mathbf{R}}_a\nabla_a + \dot{\mathbf{R}}_b\nabla_b = \tfrac{1}{2}(\dot{\mathbf{R}}_a - \dot{\mathbf{R}}_b)(\nabla_a - \nabla_b)$$

$$+ \tfrac{1}{2}(\dot{\mathbf{R}}_a + \dot{\mathbf{R}}_b)(\nabla_a + \nabla_b),$$

$$\dot{\mathbf{R}}_a - \dot{\mathbf{R}}_b = \dot{\mathbf{R}}, \quad \nabla_a - \nabla_b = \nabla_R.$$

The first term of the expression in the right half has Galilean invariance, while the second does not. It is easy to show that

$$(\nabla_a + \nabla_b)_{12}^0 = (\nabla_a + \nabla_b)_{12} = \hbar^{-2}m\mathbf{D}_{12}(E_1 - E_2), \tag{13}$$

where m is the electron mass and **D** is the dipole moment of the electrons. Therefore, in the case when $E_1 - E_2 \approx 0$, the matrix element K_{12} is approximately Galilean-invariant.

It is seen from (10) that the matrix element V' consists of a real and imaginary part. With allowance for (9) and for the fact that $V = \frac{1}{2}(E_1 - E_2)_{\min} = \Delta E/2$, the square of the modulus of this matrix element is

$$|V'|^2 = [(\Delta E)^2/4 + (\hbar(\dot{\mathbf{R}}\nabla_R)^0_{12})^2]_{R=R_0} = (\Delta E)^2/4 + \hbar^2 v_k^2 k^2,$$

$$\mathbf{k} = (\nabla_R)^0_{12}, \tag{14}$$

where $\mathbf{R}$ is the radius vector joining the nuclei and v_k is the component of the velocity v in the direction of the vector $\mathbf{k}$.

We thus obtain for S instead of (12) the expression

$$S = a/v_R + bv_k^2/v_R, \qquad v_k \sim v_R \sim v = |\dot{\mathbf{R}}|, \tag{15}$$

$$a = \pi(\Delta E)^2/2\hbar|F^0_{12}|, \qquad b = 2\pi\hbar k^2/|F^0_{12}|. \tag{16}$$

From an analysis of the dependence of the probability of transition w on the velocity v we see that the curve of this dependence has two maxima at values $v_k\sqrt{ab}/v_R = \alpha v_k/v_R < \frac{1}{2}\ln 2$; at values $\alpha v_k/v_R \geq \frac{1}{2}\ln 2$, both maxima merge into one.

The total effective cross section of the process is

$$\sigma = 2\pi\int_0^{R_0} wp\,dp, \tag{17}$$

where p is the collision parameter.

We note that there are two possible types of transitions (Λ is the projection of the orbital momentum of the electrons on the axis joining the nuclei): 1) $\Delta\Lambda = 0$ (then $v_k = v_R$); 2) $\Delta\Lambda = \pm 1$ (then $v_k = v_\varphi$). For both cases we obtain one expression for the effective cross section σ:

$$\sigma = 2\pi R_0^2\int_0^1 2e^{-A/\eta - B\eta}(1 - e^{-A/\eta - B\eta})\,\eta\,d\eta, \tag{18}$$

and if $\Delta\Lambda = 0$, then

$$A = \alpha v_0/v, \qquad B = \alpha v/v_0; \tag{16a}$$

while if $\Delta\Lambda = \pm 1$, then

$$A = \alpha(v_0/v + v/v_0), \qquad B = -\alpha v/v_0. \tag{16b}$$

Here

$$v_0 = \sqrt{a/b} = \Delta E/2\hbar k, \quad \alpha = \pi\Delta Ek/|F^0_{12}|.$$

The parameters A and B, generally speaking, vary within the limits $0 \leq A \leq +\infty$ and $-\infty < B < +\infty$. In addition, $A + B > 0$ when $\Delta\Lambda = \pm 1$.

We have calculated the cross section of the process by means of formula (18) for those values of the parameter α, at which the curves $w = w(v)$ have two maxima, i.e., $\alpha < \frac{1}{2}\ln 2$. In these cases the cross section $\sigma(v)$ also has two maxima with respect to the velocities; the two maxima are equal when $\Delta\Lambda = 0$ while the second maximum (at the higher velocity) is smaller than the first when $\Delta\Lambda = \pm 1$. The smaller α, the more separated are these maxima. It is obvious that when $\alpha \ll 0.3$ the first maximum in (15) is due to the first term at low velocities. This first term, as we have seen, is due to the minimal stationary splitting of the levels (spin-orbit and polarization effects). The second maximum is due to the second term in the expression for V′ at high velocities. This term takes into account the dependence of the wave functions of the electrons on the time through the parameters (radius-vectors of the nuclei) i.e., it includes the interaction between the orbit and the motion of the nuclei.

When $\alpha \ll 0.01$ the two maxima are so separated, that in calculating the cross sections with these values of α, it is possible to consider in practice, for each of the maxima, either the first or the second terms in the expression (14) respectively.

References 4 — 6 consider several charge-exchange processes. In calculating the cross sections of these inelastic processes as functions of the velocity, these references took into account this dependence only in the first term of (14). Consequently the curves given there have only one maximum.

However, if α reaches a value 0.1, $\sigma(v)$ varies significantly even in the region of the first maximum. In half of the cases calculated by Bates, Moiseiwitsh, and Boyd,[4] α is found to be of order 0.1. For illustration, we give the plots of $\sigma(v)$ for $\alpha = 0.1$ and $\Delta\Lambda = \pm 1$ and 0 (Fig. 1) and for $\alpha = 0.25$ and $\Delta\Lambda = 0$ (Fig. 2). In the last case we see that $\sigma(v)$ is approximately constant in the interval $0.6\,v_0 < v < 6\,v_0$.

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BREMSSTRAHLUNG FROM A DISTRIBUTED PROTON

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An expression is obtained for the form factor of a proton emitting bremsstrahlung radiation as a result of diffraction scattering in a nuclear field.

DIFFRACTION radiation of photons by fast particles of spin $\frac{1}{2}$ has been considered in the work of Akhiezer.[1] Sitenko, who calculated the effective cross section for radiation during the scattering of protons, took into account the Coulomb interaction.[2] As shown in these papers, however, a consistent discussion requires that the anomalous magnetic moment and "diffuseness" of the proton should be taken into account.

This is all the more important in light of experimental investigations of the scattering of fast electrons on protons conducted by Hofstadter's group.[3] As a result of this work, it was explained, in particular, that for large momenta of the recoil proton it is necessary to take into account the interaction of the proton with the mesic vacuum. It was shown that, for the calculation of this interaction, the operator γ_μ in the expression for the proton current density $u_2\gamma_\mu u_1$ should be replaced by the vertex operator[4]

$$\Gamma_\mu = a(q^2)\gamma_\mu + \frac{ib(q^2)}{2M}\gamma_\mu\hat{q}, \tag{1}$$

where q is the difference of the 4-momenta of the proton in the initial and final states; $a(q^2)$ and $b(q^2)$ are real invariant functions, the first of which describes the charge distribution and the Dirac magnetic moment of the proton, and the second the distribution of its anomalous moment.

Using the same method, we have calculated in the present work the influence of the proton structure and anomalous magnetic moment on the bremsstrahlung from the proton for diffraction scattering in the field of a nucleus. It is assumed that the form of "diffuseness" of the proton does not depend on the character of the electromagnetic process and is completely determined by the value of the proton recoil.

The matrix element of the bremsstrahlung of ultrarelativistic protons in a central field has the form

$$S_{i\to f} = -\frac{2\pi ie}{\sqrt{2\omega}}\int \bar{\psi}_{p'}^{(-)}(\mathbf{r})\,\gamma_\mu\psi_p^{(+)}(\mathbf{r})\,e_\mu \exp(-i\mathbf{kr})\,d\mathbf{r}. \tag{2}$$

In accordance with (1), we replace γ_μ in (2) by

$$A\gamma_\mu - (iB/2M)\,\gamma_\mu\hat{k}. \tag{3}$$

The minus sign in front of the second term corresponds to the radiation of a photon by the proton. In the general case, A and B are functions of the invariants

$$E\omega - \mathbf{pk}, \qquad E'\omega - \mathbf{p'k}, \qquad EE' - \mathbf{pp'} - M^2.$$

If in the radiation process the proton acceleration is not too great and, consequently, the energies of the radiated photons are not large, one may neglect the quantities $(E\omega - \mathbf{p}\cdot\mathbf{k}) - (E'\omega - \mathbf{p}\cdot\mathbf{k})$ and $EE' - \mathbf{p}\cdot\mathbf{p}' - M^2$ in comparison with M^2. Then A and B will depend only on $E\omega - \mathbf{p}\cdot\mathbf{k}$ and will coincide with the functions a and b describing the structure of the protons in elastic scattering of electrons on protons.* Carrying out the calculation of the effective bremsstrahlung cross section, we obtain

$$d\sigma = \frac{e^2}{4\pi^3}\frac{E'}{E}\left|\frac{RJ_1(MR\,|\mathbf{x}+\mathbf{y}|)}{|\mathbf{x}+\mathbf{y}|} + \frac{2in}{|\mathbf{x}+\mathbf{y}|}\int_R^\infty \exp\{2i\,[\eta(\rho) - \eta(R)]\}\,J_1(MR\,|\mathbf{x}+\mathbf{y}|\,\rho)\,d\rho\right|^2 \left\{\left(\frac{\mathbf{x}}{1+x^2} + \frac{\mathbf{y}}{1+y^2}\right)^2 + \frac{\omega}{2EE'}\frac{(\mathbf{x}+\mathbf{y})^2}{(1+x^2)(1+y^2)}\right\} F^2\,\frac{d\omega}{\omega}\,d\mathbf{x}d\mathbf{y},$$

$$F^2 = A^2\left[1 + \mu\frac{\omega}{2E}(1+x^2) - \frac{\mu^2\omega^2}{8E^2}\left(1+x^2\right)^2\right], \tag{4}$$

where $\mu = B/A$, the remaining notation being the same as in reference 2. Expression (4) differs from Sitenko's formula by a factor F^2. The form of the factor F^2 corresponds to the conclusions of Landau and Pomeranchuk, who base themselves on general considerations.[5] It should be noted that formula (4) ceases to be valid for $E\omega - \mathbf{p}\cdot\mathbf{k} \sim M^2$.

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*They will differ only in their arguments: q^2 in the case of elastic scattering and $2pk$ in the case of bremsstrahlung.

The same factor F^2, of course, enters into the formula relating the bremsstrahlung cross section and the elastic cross section obtained by Sitenko.[2]

In a similar way, the influence of the proton "structure" on the process of diffraction production of electron — positron pairs in the scattering of the proton in a central field may be taken into account.

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ON THE THEORY OF THE OPTICAL ANISOTROPY OF ATOMIC NUCLEI

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The elastic scattering of gamma rays by oriented nuclei is considered. The possibility of an experimental study of nuclear optical anisotropy parameters, including the vector electric dipole polarizability, of nuclei is discussed.

IN earlier papers by one of the present authors[1,2] the concepts of molecular optics (see Placzek's monograph, for example) were extended to photonuclear reactions. It was noted[1] that an induced dipole electric moment in a nucleus may depend on the orientation of the nuclear spins with respect to the electric field. Certain effects were also discussed, whose experimental investigation would enable us to establish the existence of optical anisotropy in nuclei and to measure its basic parameters.[2] The elastic scattering of gamma rays by nuclei was considered in particular, but effects associated with the possible existence of vector polarizability were neglected without justification.

The present note considers the elastic scattering of gamma rays by nuclei, including the previously uninvestigated case of scattering by oriented nuclei. Investigation of the latter effect would, in principle, enable us to measure the vector polarizability.

The elastic scattering of dipole gamma rays is described by the following R matrix:

$$(m\boldsymbol{\lambda}\,|\,R\,|\,m'\boldsymbol{\lambda}') = \Big\{R'\delta_{ij}\delta_{mm'} + \frac{3R^T}{J(2J-1)}\Big(m\,\Big|\,\tfrac{1}{2}(\hat{J}_i\hat{J}_j+\hat{J}_j\hat{J}_i) - \tfrac{1}{3}J(J+1)\delta_{ij}\,\Big|\,m'\Big) + R^V J^{-1} e_{ijl}(m\,|\,J_l\,|\,m')\Big\}\lambda_i'^{*}\lambda_j, \tag{1}$$

where m,m′ are the projections of the nuclear spin on the $\mathbf{e}_3$ axis; λ, λ′ are the photon polarization vectors before and after scattering, respectively; $\hat{J}_i$ is the nuclear spin projection operator; $R' \equiv R^S - T$; $T = (\omega/c)^3 e^2 Z^2/2\pi AM\omega^2$ is the amplitude of Thomson scattering by a nuclear charge Z;

$$R^S = (\omega/c)^3 c^S/2\pi, \quad R^T = (\omega/c)^3 c^T/2\pi, \quad R^V = (\omega/c)^3 c^V/2\pi, \tag{1a}$$

where c^S, c^T and c^V are the scalar, tensor and vector polarizability, respectively; e_{ijl} is a completely antisymmetric third-rank tensor ($e_{123} = 1$).

The contribution of the vector polarizability to the absorption cross section is

$$\sigma^V = \frac{16\pi^2 m}{(\omega/c)^2 J}\,\mathrm{Re}\,R^V\cdot\mathrm{Im}\,(\boldsymbol{\lambda}_1^{*}\boldsymbol{\lambda}_2), \tag{2}$$

which differs from zero only in the case of elliptic (or circular) gamma-ray polarization. The results for the absorption cross section given in reference 2 pertain to unpolarized photons and are not affected by the vector polarizability.

The cross section for elastic scattering of gamma rays by unoriented nuclei with vector polarizability taken into account is

$$\frac{d\sigma^{\text{H}}}{d\Omega} = \frac{2\pi^2c^2}{\omega^2}\Big\{|R'|^2[1+(\mathbf{k}'\mathbf{k})^2] + \frac{(J+1)(2J+3)}{5J(2J-1)}\Big|\frac{R^T}{2}\Big|^2[13+(\mathbf{k}'\mathbf{k})^2] + \frac{J+1}{3J}|R^V|^2[3-(\mathbf{k}'\mathbf{k})^2]\Big\}, \tag{3}$$

where **k** and **k′** are the unit wave vector of the photon before and after scattering, respectively.

Equation (3) shows that investigation of elastic scattering by unoriented nuclei enables us to estimate only the combined effect of the tensor and vector polarizabilities. Information concerning all three parts of the polarizability can, in principle, be obtained by investigating the angular dependence of $d\sigma^{\text{H}}/d\Omega$ or by considering $d\sigma^{\text{H}}/d\Omega$ together with the cross section for absorption by both oriented and unoriented nuclei. These procedures require the solution of a system of equations; the experimental errors could increase as a result of several algebraic operations and thus greatly hinder detection of the effects in question. In connection with the foregoing it is of interest to consider the elastic scattering of gamma rays by oriented nuclei.

The cross section for the scattering of unpolarized photons by a nucleus with spin projection m on the $\mathbf{e}_3$ axis is

$$\frac{d\sigma}{d\Omega}=\frac{d\sigma^{\mathrm{H}}}{d\Omega}+\frac{2\pi^2c^2}{\omega^2}\Bigg\{\left|\frac{3R^T}{2J\,(2J-1)}\right|^2\left(J^2+J+\frac{5}{2}\right)(m^2-\overline{m^2})$$

$$-\frac{5}{2}(m^4-\overline{m^4})\left[k_3'^2+k_3^2+k_3'^2k_3^2-\frac{11}{15}-\frac{2}{15}(\mathbf{k}'\mathbf{k})^2\right]$$

$$+\left|\frac{3R^T}{2J\,(2J-1)}\right|^2\ [(-8J^2-8J+5)(m^2-\overline{m^2})$$

$$+10(m^4-\overline{m^4})]\left[2k_3'^2k_3^2-k_3'k_3(\mathbf{k}'\mathbf{k})-\frac{2}{15}+\frac{1}{15}(\mathbf{k}'\mathbf{k})^2\right]$$

$$-6\,\mathrm{Re}\left(\frac{3R^{T*}}{2J\,(2J-1)}\,R'\right)(m^2-\overline{m^2})\Big[k_3'^2+k_3^2-k_3'k_3\,(\mathbf{k}'\mathbf{k})$$

$$-\frac{2}{3}+\frac{1}{3}(\mathbf{k}'\mathbf{k})^2\Big]-6\,\mathrm{Im}\left(\frac{3R^{T*}}{2J\,(2J-1)}\,\frac{R^V}{J}\right)$$

$$\times(m^2-\overline{m^2})(k_3'^2-k_3^2)+\left|\frac{R^V}{J}\right|^2(m^2-\overline{m^2})(\mathbf{k}'\mathbf{k})[3k_3'k_3$$

$$-(\mathbf{k}'\mathbf{k})]+2\mathrm{Re}\left(R'^*\,\frac{R^V}{J}\right)\,m\,(\mathbf{k}'\mathbf{k})\,[\mathbf{k}'\times\mathbf{k}]_3$$

$$+2\,\mathrm{Re}\left(\frac{3R^{T*}}{2J\,(2J-1)}\,\frac{R^V}{J}\right)$$

$$\times m\,[5m^2-(3J^2+3J-1)]\,k_3'k_3\,[\mathbf{k}'\times\mathbf{k}]_3$$

$$-2\,\mathrm{Re}\left(\frac{3R^{T*}}{2J\,(2J-1)}\,\frac{R^V}{J}\right)$$

$$\times\,m\,(m^2-\overline{m^2})\,(\mathbf{k}'\mathbf{k})[\mathbf{k}'\times\mathbf{k}]_3\Bigg\},\qquad(4)$$

where

$$\overline{m^n}=\frac{1}{(2J+1)}\sum_{m=-J}^{m=J}m^n$$

is the mean value of the n-th power of the spin projection in the case of a nucleus freely oriented in space.

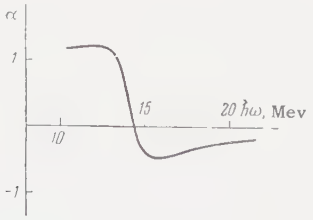
The tensor and vector polarizabilities are revealed most strongly in the azimuthal scattering asymmetry. The evaluation of

$$\alpha=\frac{\left[\frac{d\sigma}{d\Omega}(k_3'=0,\ k_3=0,\ \mathbf{k}'\mathbf{k}=0)-\frac{d\sigma}{d\Omega}(k_3'=1,\,k_3=0,\,\mathbf{k}'\mathbf{k}=0)\right]}{(d\sigma^{\mathrm{H}}/d\Omega)\,(\mathbf{k}'\mathbf{k}=0)}$$

where m = J and the expressions for the tensor and scalar polarizabilities given in reference 2 are used as well as the giant resonance parameters for ${}_{73}\mathrm{Ta}^{181}$ given in reference 4, yields

$$\alpha\,(\hbar\omega_1=12.5\ \mathrm{Mev})=1.2;\qquad\alpha\,(\hbar\omega_2=15.5\ \mathrm{Mev})=-0.4.$$

The energy dependence of α for Ta^{181} is shown in the figure, assuming $R^V = 0$. A comparison of our results with estimates of the influence of tensor polarizability on photoabsorption by oriented nuclei indicates that the tensor polarizability is considerably more important in the case of scattering by oriented nuclei.



The existence of vector polarizability can be established by measuring the difference $d\sigma(\mathbf{k},\mathbf{k}')/d\Omega-d\sigma(\mathbf{k}',\mathbf{k})/d\Omega$. In the region of dipole resonance frequencies the vector polarizability of an axisymmetric nucleus can be put into the form

$$c^V=\frac{K}{J+1}\sum_r\frac{\hbar\omega+i\gamma_r/2}{(E_r-E_0)^2-\hbar^2\omega^2-i\hbar\omega\gamma_r}$$

$$\times\,[(0\,|\,d_\eta\,|\,r)(r\,|d_\xi\,|\,0)-(0\,|\,d_\xi\,|\,r)(r\,|d_\eta|0)],$$

where K is the projection of the nuclear spin on its ζ symmetry axis; ξ, η, ζ is a right-handed system of orthogonal axes attached to the nucleus; **d** is the dipole moment operator. It is easily seen that if the nuclear energy levels E_r merge, so that all of them can be effectively replaced by one or two levels, we have $c^V = 0$. Thus vector polarizability is a manifestation of the fine structure of giant resonance.

An indication of the possible presence of vector polarizability follows from the condition

$$\sum_r i\,(E_r-E_0)[(0\,|\,d_\xi\,|\,r)(r\,|\,d\ \ |\,0)-(0\,|\,d_\eta\,|\,r)(r\,|\,d_\xi\,|\,0)]$$

$$=\frac{e^2\hbar^2}{M}\left(\frac{N^2}{A^2}\sum_p m_p+\frac{Z^2}{A^2}\sum_n m_n\right),\qquad(5)$$

where m_p, m_n are the projections of the proton and neutron orbital moment, respectively, on the nuclear symmetry axis.* The ratio of the right-hand part of (5) to the corresponding sum for scalar polarizability is of the order 1/A. On the basis of the foregoing the vector polarizability would be small in heavy nuclei, where giant resonance is unlikely to have a fine structure.

The authors wish to thank Dr. E. Fuller for discussions of a number of questions concerning the optical anisotropy of nuclei. Dr. Fuller also privately communicated to us the advisability of investigating effects associated with vector polarizability.

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*Effects associated with exchange forces are not being taken into account here.

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PION-NUCLEON SCATTERING AT LOW ENERGIES, I

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Integral equations for πN-scattering partial waves in the low-energy region are derived on the basis of the Mandelstam representation. The kernels of the equations contain the lowest $\pi\pi$- scattering phases.

1. INTRODUCTION

THE Mandelstam double dispersion representation[1] for a two-particle Green's function makes it possible to consider the matrix elements of the interaction processes corresponding to this Green's function as different limiting values of the same analytic function of two complex variables. This representation leads to simple dispersion relations in one variable, including the usual dispersion relations in the energy for arbitrary values of momentum transfer. Using the unitarity condition for the imaginary parts of the different amplitudes, we arrive at the possibility of obtaining a system of relations between the amplitudes of the various processes.

The unitasity condition, which utilizes the expansion in a complete system of intermediate states, introduces an infinite set of corresponding amplitudes. Confining ourselves in the unitarity condition to the lowest two-particle mass state, we arrive at a system of equations for the matrix elements of two-particle processes. The two-particle approximation does not lead to an error in the imaginary parts up to the threshold of the next mass state, and consequently the integrated contribution of the latter can be considered small in the low-energy region. This approximation can be considered as formulated by Chew's statement that "the behavior of the analytic function in a small region is determined essentially by the nearest singularities."[2]

The program developed here yields equations for two-particle amplitudes in the low-energy region. It is clear that an important role is played here by the scattering amplitudes of the lightest particles. Neglecting electromagnetic effects, we obtain for "ordinary" strongly-interacting particles the sequence of processes shown in Fig. 1.

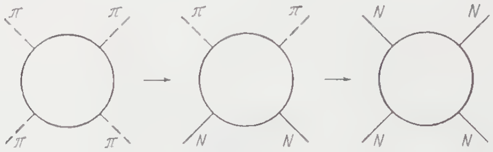

FIG. 1.

The scheme of Fig. 1 denotes, for example, that the system of equations for the processes $\pi N \rightarrow \pi N$ and $\pi\pi \rightarrow N\bar{N}$ contains the amplitudes of the process $\pi\pi \rightarrow \pi\pi$. For $\pi\pi$ scattering, to the contrary, the system of equations will be closed. Thus, the process $\pi\pi \rightarrow \pi\pi$ becomes very important in the theory of strong interactions, and serves as its "starting point."

The equations for the $\pi\pi$-scattering process were obtained by Chew and Mandelstam.[3] These equations, like the Chew-Low equations, are singular nonlinear integral equations. We know[4] that such equations have sets of solutions. Even the determination of some definite branch of these solutions is a complicated task, which can be accomplished only with the aid of high speed electronic computers. At the present time, a particular "adiabatic" solution of the Chew-Mandelstam equations has been obtained,[5] with the s wave predominating.

The block $\pi\pi$NN of Fig. 1, which describes the processes

$$\text{I.}\quad \pi + N \rightarrow \pi' + N,$$

$$\text{II.}\quad \pi' + N \rightarrow \pi + N,$$

$$\text{III.}\quad \pi + \pi' \rightarrow N + \bar{N},$$

was considered by Frazer and Fulco[6] as applied to the reaction III. They obtained for the amplitudes of process III equations, that contain the πN-scattering amplitudes and the $\pi\pi$-scattering phase shifts.

Attempts to investigate the $\pi\pi$NN block as applied to πN scattering were undertaken by McDowell.[7] He considered the analytic properties of the partial amplitudes of meson-nucleon scattering* in the complex plane of the variable s — the square of the total energy of the meson + nucleon system [see (3.1) below] and established the kinematic complex singularities due to the inequality of the masses. These singularities make it difficult to obtain the integral equations. Recently, however, Chou Hung-Yüan (private communication) was able to carry the corresponding derivations through to conclusion and obtained a system of integral equations for the partial-scattering waves and for the process $\pi\pi \rightarrow N\bar{N}$.

In the present paper we investigate the analytical properties of the πN scattering amplitudes in the complex plane of the square of the momentum, variable q^2, in the c.m. system for a fixed scattering angle. This approach is analogous to that of Cini-Fubini-Stanghellini[8] to the $NN \rightarrow NN$ process.

In the plane of the variable q^2, the kinematic singularities have the form of a finite cut on the real axis and can be eliminated by considering the corresponding symmetrized and antisymmetrized expressions.

Confining ourselves in the unitarity conditions for reaction III to the two-meson state, we leave in the $\pi\pi \rightarrow \pi\pi$ amplitudes only the s and p phase shifts. This allows us to obtain for the amplitudes of processes I and II an integral equation that contains only the aforementioned two phase shifts δ_0 and δ_1 for $\pi\pi$ scattering. The use of the cited Chew principle allows us to neglect the non-physical cuts due to reaction II and to obtain integral equations for the partial waves of πN scattering. Naturally, these equations can have a meaning only in the region of low energies.

Section 2 contains a summary of the formulas for the matrix elements of processes I, II, and III in terms of the invariant coefficients of the two-particle Green's function, and also the unitarity conditions for the partial waves.

In Sec. 3 we choose the functions and the variable for the analytic continuation. This continuation, as well as the elimination of the kinematic cuts, is realized in Sec. 4. Next (Sec. 5) we analyze the nearest part of the non-physical cut of reaction III, by means of an approximate unitarity condition which contains only the s and p phase shifts of the $\pi\pi$ scattering. With the aid of the Muskhelishvili method we obtain for the amplitudes of πN scattering integral equations that contain integrals only over the physical regions of reactions I and II. The $\pi\pi$-scattering phase shifts enter into the kernels of these equations. In Sec. 6 we go over to the partial waves in the final equations.

The foregoing brief survey of papers on the use of the Mandelstam representation is incomplete. Mention should be made of the very important paper by Ter-Martirosyan,[9] in which the unitarity conditions in the two-particle approximation, subject to an imposed crossing symmetry, are used to obtain equations for the spectral functions of the Mandelstam representation. An interesting method for the approximate reduction of the double Mandelstam representation to a sum of one-dimensional representations was proposed by Cini and Fubini.[10] Their technique makes it possible to simplify the derivation of the equations for the partial waves.

2. MATRIX ELEMENTS, UNITARITY CONDITIONS

The matrix elements of processes I, II, and III, described by the block $\pi\pi NN$, are represented in the form

$$< f\,|\,S-1\,|\,i >$$
$$= \frac{i}{4\pi^2}\,\delta\,(p_1 + p_2 + q_1 + q_2)\sqrt{\frac{M^2}{4p_1^0p_2^0q_1^0q_2^0}}\,\bar{u}Tu, \qquad (2.1)$$

p_1 and p_2 are the 4-momenta of the nuclei while q_1 and q_2 are those of the mesons. The two-particle Green's function T has the following structure:

$$T = A + \tfrac{1}{2}\,(\hat{q}_1 - \hat{q}_2)\,B, \qquad T = \delta_{\rho\rho'}T^{(+)} + \tfrac{1}{2}\,[\tau_\rho, \tau_{\rho'}]T^{(-)} \qquad (2.2)$$

The notation in (2.1) and (2.2) is standard; the Feynman metric is used: $\hat{q} = q^0\gamma_0 - \gamma\mathbf{q}$, $\gamma_0^2 = 1$, and $\gamma_\alpha^2 = -1$.

The matrix elements $\bar{u}Tu$ coincide, accurate to within a factor $M/4\pi W$ (W is the total energy of the process), with the 'spiral states' of Jacob and Wick. For the scattering process, these states have the form

$$f_{++} = f_{--} = \cos\frac{\theta}{2}\,(f_1 + f_2) = \cos\frac{\theta}{2}\,\frac{MA + (p^0W - M^2)\,B}{4\pi W},$$
$$f_{+-} = -f_{-+} = \sin\frac{\theta}{2}\,(f_1 - f_2) = \sin\frac{\theta}{2}\,\frac{p^0A + Mq^0B}{4\pi W}. \qquad (2.3)$$

Here, as in (2.6) below, the azimuth angles are assumed equal to zero; θ is the scattering angle in the c.m.s. and f_1 and f_2 are expressed in terms of the partial amplitudes:

$$f_1 = \sum_l \{f_{l,+}P'_{l+1}(\cos\theta) - f_{l,-}P'_{l-1}(\cos\theta)\},$$
$$f_2 = \sum_l (f_{l,-} - f_{l,+})\,P'_l(\cos\theta), \qquad (2.4)$$

$$f^J_{l,\pm} = q^{-1}\sin\delta^J_{l,\pm}\exp\,(i\delta^J_{l,\pm}). \qquad (2.5)$$

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*McDowell actually considered the process $K + N \rightarrow K + N$, but the kinematic singularities of this process are analogous to those of the process $\pi + N \rightarrow \pi + N$.

The spiral states for the process III will be (cf. reference 6)

$$J_{++} = J_{--} = \frac{i}{8\pi}\left\{-\frac{pA}{p^0} + \frac{qM\cos\theta_3}{p^0}B\right\},$$

$$J_{+-} = -J_{-+} = \frac{q\sin\theta_3}{8\pi}B. \qquad (2.6)$$

Here θ_3 is the angle between the vectors **p** and **q**. The expansion in the partial waves has the form

$$J_{++} = \frac{1}{pp^0}\sum_l\left(l+\frac{1}{2}\right)(pq)^l f^l_+ P_l(\cos\theta_3), \qquad (2.7)$$

$$J_{+-} = q\sum_l\frac{l+{}^1/_2}{\sqrt{l(l+1)}}(pq)^{l-1}f^l_- P^{(1)}_l(\cos\theta_3). \qquad (2.8)$$

If we introduce into these formulas the isotopic indices + and −, then the summation goes only over even l for the + sign and over odd l for the − sign.

We consider the antihermitian part of the third process, confining ourselves to the two-meson term in the sum over the complete system of functions. This system can be represented in the form

$$\operatorname{Im}J^{(\pm)}_{\lambda\mu} = \frac{q}{64\pi^2q^0}\int d\Omega_{\mathbf{q}'}J^{(\pm)}_{\lambda\mu}(\mathbf{p}, -\mathbf{p};\ \mathbf{q}',$$
$$-\mathbf{q}')\,\Pi^{(0,1)*}(\mathbf{q}', -\mathbf{q}';\ \mathbf{q}, -\mathbf{q}). \qquad (2.9)$$

Here Π^0 and Π^1 are the $\pi\pi$ scattering amplitudes with total isotopic spin 0 and 1, respectively [see, for example, formula (2.8) in the paper by Chew and Mandelstam[3]]. It is seen from (2.9) that $J^{(\pm)}$ are the amplitudes of process III, with total isotopic spins 0 and 1.

We shall consider below the analytic continuation of (2.9) in the non-physical region of small q^2. In this case we can confine ourselves to the s and p waves in the amplitude Π. Expanding (2.9) in partial waves and expressing the results through A and B, we obtain in this approximation

$$\operatorname{Im}A^{(+)} = -4\pi p^{-2}f^{0(+)}_+e^{-i\delta_0}\sin\delta_0,$$

$$\operatorname{Im}B^{(-)} = 6\sqrt{2}\,\pi f^{1(-)}_-e^{-i\delta_1}\sin\delta_1.$$

$$\operatorname{Im}A^{(-)} = 12\,\pi qp^{-1}\cos\theta_3\{2^{-1/2}f^{1(-)}_- - f^{1(-)}_+\}e^{-i\delta_1}\sin\delta_1,$$

$$\operatorname{Im}B^{(+)} = 0. \qquad (2.10)$$

Here δ_0 is the s phase with J = 0 and δ_1 the p phase for J = 1.

3. KINEMATICS; CHOICE OF VARIABLES AND FUNCTIONS FOR THE ANALYTIC CONTINUATION

We introduce the ordinary invariants

$$s = (p_1+q_1)^2, \quad \bar{s} = (p_1+q_2)^2, \quad t = (p_1+p_2)^2,$$
$$s + \bar{s} + t = 2(\mu^2 + M^2), \qquad (3.1)$$

which have the following form in c.m.s. of the reactions I and III

$$s = M^2 + \mu^2 + 2q^2 + 2\sqrt{q^2+\mu^2}\sqrt{q^2+M^2},$$

$$\text{I} \quad \bar{s} = M^2 + \mu^2 - 2q^2\cos\theta - 2\sqrt{q^2+\mu^2}\sqrt{q^2+M^2}, \qquad (3.2)$$

$$t = -2q^2(1-\cos\theta);$$

$$s = -2q_3^2 + M^2 - \mu^2 + 2p_3q_3\cos\theta_3,$$

$$\bar{s} = -2q_3^2 + M^2 - \mu^2 - 2p_3q_3\cos\theta_3,$$

$$\text{III} \quad t = 4(q_3^2 + \mu^2) = 4(p_3^2 + M^2). \qquad (3.3)$$

The analytic properties of the invariant scalar functions $A^{(\pm)}$ and $B^{(\pm)}$, introduced in (2.2), are described by the Mandelstam representation[1] [see formula (2.12) of that paper]. It is important for what is to follow that the representations for $B^{(\pm)}$ contain the pole terms

$$B^{(\pm)}(s, \bar{s}, t) = \frac{g^2}{M^2 - s} \pm \frac{g^2}{M^2 - \bar{s}} + \cdots, \qquad (3.4)$$

while the representations for $A^{(\pm)}$ do not contain these terms.

We choose for the analytic continuation the four functions Φ:

$$\Phi(s, \bar{s}, t) = A^{(+)}, \ \alpha, \ \beta, \ B^{(-)};$$

$$\alpha = A^{(-)}/(s - \bar{s}), \qquad \beta = B^{(+)}/(s - \bar{s}). \qquad (3.5)$$

All these functions have the property

$$\Phi(s, \bar{s}, t) = \Phi(\bar{s}, s, t). \qquad (3.6)$$

In addition, they diminish at infinity not slower than A and B. Division by $s - \bar{s}$ does not introduce any new singularities.

We shall consider the functions Φ as applied to reaction I. We fix the scattering angle as $\cos\theta = c$, i.e., we consider the analytical properties in the variable q^2. It is now convenient to rewrite (3.2) in the form

$$s = s(q^2) = R(q^2) + D(q^2) + 2K(q^2),$$

$$\bar{s} = \bar{s}(q^2) = R(q^2) - D(q^2) - 2K(q^2),$$

$$t = -2q^2(1-c), \qquad (3.7)$$

where

$$R(q^2) = M^2 + \mu^2 + q^2(1-c), \quad D(q^2) = q^2(1+c),$$
$$K(q^2) = \sqrt{(q^2+M^2)(q^2+\mu^2)}. \qquad (3.8)$$

4. ELIMINATION OF THE KINEMATIC CUTS AND THE CAUCHY THEOREM

The functions $\Phi(s, \bar{s}, t) = \Phi(q^2)$ for $\cos\theta = c$), considered in the complex plane of q^2, have the following singularities: 1) a cut $-\infty < q^2 < -2\mu^2/(1-c) = -a_3$, connected with the denominator $t' - t$ in the spectral formula of Mandelstam; 2) a cut $0 < q^2 < \infty$ due to the denominator $s' - s$; 3) a cut $-\infty < q^2 < -a(c)$ due to $\bar{s}' - \bar{s}$

$$a(c) = \begin{cases} M^2 & 1 \geqslant c \geqslant \mu/M \\ a_2 = (M^2 + \mu^2 - 2M\mu c)(1 - c^2)^{-1}, & \mu/M \geqslant c \geqslant -1; \end{cases} \tag{4.1}$$

4) a cut $-M^2 < q^2 < -\mu^2$, due to the square root (3.8) in the dependence of s and $\bar{s}$ on q^2. We call this the kinematic cut.

In addition, the functions β and $B^{(-)}$ have poles connected with (3.4).

On changing to the variable s, the kinematic cut $-M^2 < q^2 < -\mu^2$ gives a complex cut lying on a circle (see reference 7). To get rid of this cut, we represent Φ in the following manner

$$\Phi(q^2, K) = \Phi_1(q^2) + K(q^2)\Phi_2(q^2), \tag{4.2}$$

distinctly separating the irrational dependence. The functions Φ_1 contain no irrational dependences. They can be determined as

$$\Phi_1(q^2) = \frac{1}{2}[\Phi(q^2, K) + \Phi(q^2, -K)], \tag{4.3}$$

$$\Phi_2(q^2) = \frac{1}{2}[\Phi(q^2, K) - \Phi(q^2, -K)]K^{-1}(q^2). \tag{4.4}$$

Let us explain the meaning of the function $\Phi(q^2, -K(q^2))$. According to (3.7), we have $\Phi(q^2, -K) = \Phi(s^*, \bar{s}^*, t)$, with

$$s^*(q^2) = R(q^2) + D(q^2) - 2K(q^2), \quad \bar{s}^*(q^2) = R(q^2) - D(q^2) + 2K(q^2). \tag{4.5}$$

The variables s^*, and $\bar{s}^*$ are given here in terms of the variables q^2 and c of reaction I.

The point $s^*, \bar{s}^*, t$ lies in the physical region of reaction II if (q^2, c) lies in the physical region of reaction I. The connection between these points is given by

$$q_2^2 = q^2\,\frac{M^2 + \mu^2 + q^2(1 + c^2) - 2cK(q^2)}{M^2 + \mu^2 - 2q^2c + 2K(q^2)},$$

$$c_2 = 1 - \frac{q^2}{q_2^2}(1 - c). \tag{4.6}$$

[for a geometrical interpretation of relations (4.6) see the appendix].

The functions $\Phi(q^2, -K)$, considered in the complex plane of q^2, have the following singularities: 1) a cut $0 < q^2 < \infty$ due to the denominator $\bar{s}' - \bar{s}^*$ in the spectral formula of Mandelstam; 2) a cut $-a_2 < q^2 < -M^2$ due to the same denominator when $c \geq \mu/M$; 3) a cut $-\infty < q^2 < -2\mu^2/(1-c)$ due to the denominator $t' - t$; 4) a kinematic cut $-M^2 < q^2 < -\mu^2$. In addition, the functions β and $B^{(-)}$ of the arguments q^2 and $K(q^2)$ have poles.

The functions $\Phi_1(q^2)$ have all the singularities of the functions $\Phi(q^2, K)$ and $\Phi(q^2, -K)$, with the exception of the kinematic cut.

Writing the Cauchy formulas for these and returning to the initial function $\Phi(q^2, K)$, we obtain

$$\begin{aligned}\pi\Phi(q^2, K) = {} & \int_0^{\infty} [\operatorname{Im}\Phi(q'^2, K(q'^2))\varkappa_+(q^2, q'^2) \\ & + \operatorname{Im}\Phi(q'^2, -K')\varkappa_-(q^2, q'^2)]\frac{dq'^2}{q'^2 - q^2} \\ & + \int_{-\infty}^{-M^2} [\operatorname{Im}\Phi(q'^2, K')\varkappa_+(q^2, q'^2) \\ & + \operatorname{Im}\Phi(q'^2, -K')\varkappa_-(q^2, q'^2)]\frac{dq'^2}{q'^2 - q^2} \\ & + \int_{-\infty}^{-a(c)} \frac{\operatorname{Im}\Phi(q'^2, K')\varkappa_+(q^2, q'^2)}{q'^2 - q^2}dq'^2 \\ & + \theta\left(c - \frac{\mu}{M}\right)\int_{-a_2}^{-M^2} \frac{\operatorname{Im}\Phi(q'^2, -K')\varkappa_-(q^2, q'^2)}{q'^2 - q^2}dq'^2 + \text{pole}\end{aligned}$$

terms for β and $B^{(-)}$. (4.7)

Here

$$\varkappa_\pm(q^2, q'^2) = \frac{1}{2}(1 \pm K(q^2)/K(q'^2)). \tag{4.8}$$

Figure 2 shows the integration domains in (4.7) as functions of c. It is seen from this figure that the cases of backward and forward scattering are the simplest. Thus, when $c = -1$ the non-physical cut due to reaction (2) (domain II′) goes to infinity, only the cut due to reaction III remains at $q^2 < 0$. It can be shown that in this case the integral contains the antihermitian part of the amplitude of the physical value $\cos\theta_3 = -1$.

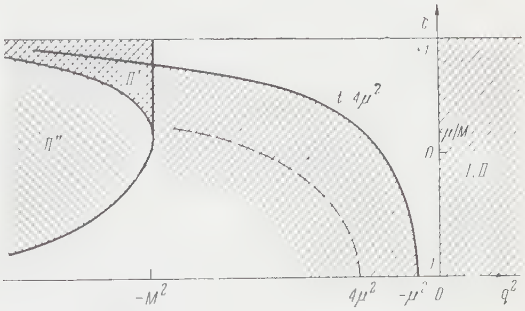

FIG. 2

For forward scattering $(c = +1)$, to the contrary, it is the cut due to reaction III that goes to infinity. In this case the antihermitian parts under the integrals over the domains II′ and II″ also depend on the physical value $\cos\theta_2 = +1$. These relations coincide exactly with usual forward dispersion relations in the energy variable E in the laboratory system of coordinates. The integration domains are as listed on the table.

	Domain of variable q^2	Domain of variable E
I	$0, \infty$	μ, ∞
II	$0, \infty$	$-(M^2+\mu^2)/2M, -\mu$
II′	$-\infty, -M^2$	$-M, -(M^2+\mu^2)/2M$
II″	$-\infty, -M^2$	$-\infty, -M$

5. INVESTIGATION OF THE CUT DUE TO REACTION III

The second integral (4.7) contains the antihermitian part of the amplitude of reaction III in the non-physical region $t < 4M^2$. We shall consider the quantities Im Φ in this region as analytic continuations of the corresponding functions from the physical region $t > 4M^2$. Their explicit expressions can be obtained from (2.2) by recalling that $s - \bar{s} = 4p_3q_3 \cos\theta_3$.

In view of the fact that in our approximation (only the s and p phase shifts of the $\pi\pi$ scattering are considered) the expressions (2.10) do not contain $\cos\theta_3$, they coincide with the numerator in the second integral of (4.7). In the more general case, when the higher-order phase shifts are considered, these expressions depend on $\cos\theta_3$ and the following connection must be used

$$2(\mathbf{p}_3\mathbf{q}_3) = \tfrac{1}{2}(s - \bar{s}) = D(q^2) + 2K(q^2). \tag{5.1}$$

It is seen from (5.1) that $\cos\theta_3$ becomes complex, but the numerator of the second integral in (4.7) remains real. In order to connect Im Φ with the functions Φ themselves, it is necessary to turn to expansions of the type (2.7) and (2.8).

These expansions are really in the parameter $s - \bar{s}$. The radius of convergence of such an expansion is determined by the singularities of the Mandelstam representation. The nearest singularities are the poles due to the single-nucleon denominators. In the approximation (2.10) these singularities are essential only for the function $B^{(-)}$. We shall consider them later.

The following singularities are due to the integrals of the double spectral representation. It can be shown that these singularities do not prevent expansion up to

$$q'^2 \sim -2.3\,\mu^2. \tag{5.2}$$

Therefore the expansions (2.7) and (2.8) are correct only in the region (5.2). We assume that the terms of these expansions diminish rapidly and confine ourselves to the first terms. This yields

$$\operatorname{Im} A^{(+)} = A^{(+)} e^{-i\delta_0} \sin\delta_0, \qquad \operatorname{Im}\alpha = \alpha e^{-i\delta_1}\sin\delta_1. \tag{5.3}$$

Let us turn to Eq. (4.7) and let us compare the effect due to different non-physical contributions from the interval $q'^2 < 0$. Assuming henceforth a transition to the partial scattering amplitudes, i.e., averaging over $\cos\theta = c$, we should examine the roles of these cuts in the variable q^2 independently of c. It follows from Fig. 2 that the nearest cut, beginning with the point $-\mu^2$, is connected with the reaction III. For arbitrary c, it is bounded by the curve $t = 4\mu^2$.

The use of the unitarity conditions with two mesons in the intermediate state allows us to take accurate account of this cut up to the point $-4\mu^2$ (there are grounds[3,6,12] for assuming that these relations will be true also in a certain region below $-4\mu^2$). However, as noted earlier [see (5.2)], formulas (5.3) cannot be correct for values lower than $-2.3\mu^2$. Thus, it is sensible to expect that formulas (2.3) will enable us to take into account the nearest non-physical singularities in the interval $-2.3\,\mu^2 < q'^2 \leq -\mu^2$.

It is clear that we should therefore neglect completely the contributions due to the nonphysical regions II′ and II″, located behind $-M^2$. The estimates for the forward scattering, when regions II′ and II″ make up the entire non-physical contribution, have shown that their neglect at small energies, down to values of E on the order of 100 Mev, causes an error of approximately 10%. The global effect is expected to decrease after averaging over the angle. As a result we obtain the following equation for β:

$$\beta(q^2, K) = \frac{g^2}{(M^2 - s)(M^2 - \bar{s})} + \frac{1}{\pi}\int_0^\infty \operatorname{Im}\beta(q'^2, K')\,\frac{\varkappa_+(q^2, q'^2)\,dq'^2}{q'^2 - q^2} + \frac{1}{\pi}\int_0^\infty \operatorname{Im}\beta(q'^2, -K')\,\frac{\varkappa_-(q^2, q'^2)}{q'^2 - q^2}\,dq'^2. \tag{5.4}$$

The corresponding equations for $A^{(+)}$ and α can be simplified by returning to the functions Φ_1 and Φ_2. In our approximation we have Im $\Phi_2 = 0$ and Im $\Phi_1 = \Phi_1 e^{-i\delta}\sin\delta$ on the cut $(-2.3\mu^2, -a_3)$. If we consider the integrals over the domain $0, \infty$ to be known functions, then the equation for Φ_1 will be a linear singular equation, which can be solved by Muskhelishvili's method (see Chapter 5 of the book by Muskhelishvili,[13] and also the paper by Omnes[14]). In our case this method reduces to an examination not of $\Phi_1(q^2)$ but of the function $\Phi_1(q^2)\exp[-u(q^2)]$, where

$$u(q^2) = -\frac{1}{\pi}\int\limits_{-2.3\mu^2}^{-a_3}\frac{\delta\left(-q'^2(1-c)/2-\mu^2\right)}{q'^2-q^2}\,dq'^2, \qquad (5.5)$$

which has no singularities when $q^2 < 0$.

Solving the equation in this manner, we obtain after simple transformations

$$\Phi(q^2, K) = \frac{1}{\pi}\int\limits_0^\infty \frac{\mathrm{Im}\,\Phi_{\mathrm{I}}(q'^2)\,\varkappa_+(\delta, q^2, q'^2)}{q'^2-q^2}\,dq'^2$$

$$+\frac{1}{\pi}\int\limits_0^\infty \frac{\mathrm{Im}\,\Phi_{\mathrm{II}}(q'^2)\,\varkappa_-(\delta, q^2, q'^2)}{q'^2-q^2}\,dq'^2, \quad \Phi_{\mathrm{I,\,II}} = \Phi(q^2, \pm K), \qquad (5.6)$$

$$\varkappa_\pm(\delta, q^2, q'^2) = \tfrac{1}{2}\left[\exp\left[u(q^2)-u(q'^2)\right] \pm K(q^2)/K(q'^2)\right]. \qquad (5.7)$$

We note that our approximation (5.3) is equivalent to the condition $\Phi_2(q^2) = 0$ when $q^2 < -a_3$.

Let us return to the function $B^{(-)}$, for which the Mandelstam representation contains pole terms which make impossible an expansion in the argument $s - \bar{s}$ in the domain of the reaction III. We shall assume that the large values of the real parts of the higher partial waves f^l $(l \geq 3)$ in the expansion (2.8) are due entirely to these pole terms. (Analogous considerations were used by Okun' and Pomeranchuk[15] in an analogous analysis of the higher partial waves in NN scattering.) This enables us to write

$$\mathrm{Im}\,B^{(-)} = \{B^{(-)} - g^2(\Delta - \langle\Delta\rangle_1)\}\,e^{-i\delta_1}\sin\delta_1. \qquad (5.8)$$

Here Δ denotes the pole term, and $\langle \Delta \rangle_1$ its first term in the expansion of the angle of reaction III,

$$\langle\Delta\rangle_1 = \frac{1}{2}\int\limits_{-1}^{+1}\frac{2M^2-s-\bar{s}}{(M^2-s)(M^2-\bar{s})}\,d\cos\theta_3$$

$$= \frac{1}{2Z(t)}\ln\frac{R(t)+2Z(t)}{R(t)-2Z(t)},$$

$$Z(t) = \sqrt{(t/4-\mu^2)(t/4-M^2)}, \qquad (5.9)$$

where we choose that branch of the function $\langle \Delta \rangle_1$ which is real in the physical region of reaction I.

Carrying out the transformation with e^{-u}, we obtain

$$B^{(-)} = g^2(\Delta - \langle\Delta\rangle_1)$$

$$+\frac{1}{\pi}\int\limits_0^\infty\frac{\varkappa_+(\delta, q^2, q'^2)\,\mathrm{Im}\,B_{\mathrm{I}}^{(-)} + \varkappa_-(\delta, q^2, q'^2)\,\mathrm{Im}\,B_{\mathrm{II}}^{(-)}}{q'^2-q^2}\,dq'^2. \qquad (5.10)$$

In our approximation, the part of the function $B^{(-)}$ which is antisymmetrized in K is equal to the antisymmetrized part of the pole terms, i.e., $B_2^{(-)} = g^2\Delta_2$. We note further that an analogous method of eliminating the difficulties connected with the poles was recently proposed by Cini and Fubini.[10]

6. TRANSITION TO THE PARTIAL SCATTERING AMPLITUDES

In order to convert relations (5.4), (5.6), and (5.10) into a closed system of equations, it remains for us to use the unitarity relations and to go over to the partial scattering amplitudes. From (2.3), with allowance for the fact that

$$s - \bar{s} = 2(1+\cos\theta)\,q^2 + 4q^0p^0,$$

we obtain expression for the functions Φ in terms of the variables and amplitudes of the reaction I. The $f_{1,2}^{(\pm)}$ which are contained therein depend on the arguments q^2 and $\cos\theta = c$, and are expressed with the aid of (2.4) in terms of the partial waves f_l, which according to (2.5), have the unitarity property.

Thus, we can go over to the partial amplitudes f_l either from the function Φ or from $\mathrm{Im}\,\Phi_1(q'^2)$, which enter into the integral terms of the assembly (5.4), (5.6), and (5.10). In order to go over to f_l from $\mathrm{Im}\,\Phi_{\mathrm{II}} = \mathrm{Im}\,\Phi(q'^2, -K)$, we note, firstly, that formulas (2.3) retain their form also in the variables of the reaction II. For this purpose it is necessary only to go over in these formulas to the corresponding variables in the c.m.s. of the reaction II: $\cos\theta \to \cos\theta_2$, $q \to q_2$, $q^0 \to q_2^0$, and $p^0 \to p_2^0$. On the other hand, we transform to the variables of reaction II in the arguments $\mathrm{Im}\,\Phi_{\mathrm{II}}$ with the aid of formulas (4.6).

From these we obtain

$$\cos\theta_2 = [2q_0^2 + M^2 + \mu^2 - 2cK(q_2^2)$$

$$-\Xi(q_2^2, c)]/2q_2^2(1+c),$$

$$\Xi(q_2^2, c) = \{[2q_2^2 + 2K(q_2^2) - c(M^2+\mu^2)]^2$$

$$+(1-c^2)(M^2-\mu^2)^2\}^{1/2}. \qquad (6.1)$$

Going over to the new integration variable q'^2 in the integrals that contain $\mathrm{Im}\,\Phi_{\mathrm{II}}(q'^2)$,

$$q'^2 = [\Xi(q_2'^2, c) + 2c[q_2'^2 + K(q_2'^2)] - M^2 - \mu^2]/2(1-c^2),$$

$$K(q'^2) = [2K(q_2'^2) + 2q_2'^2 - c(M^2+\mu^2-\Xi)]/2(1-c^2),$$

$$\frac{dq'^2}{dq_2'^2} = \frac{2K(q_2'^2)+M^2+\mu^2+2q^2}{\Xi(q_2'^2, c)}\,\frac{K(q'^2)}{K(q_2'^2)} = D(q_2'^2, c), \qquad (6.2)$$

we can represent these in the form

$$\int\limits_0^\infty\frac{\mathrm{Im}\,\Phi(q'^2)}{q'^2-q^2}\,\varkappa_-(\delta, q^2, q'^2)\,dq'^2$$

$$= \int\limits_0^\infty\frac{\mathrm{Im}\,\Phi(q_2'^2, c_2(q_2'^2, c))}{q'^2(q_2'^2, c)-q^2}\,D(q_2'^2, c)\,\varkappa_-(\delta, q_2'^2, c, q^2)\,dq_2'^2. \qquad (6.3)$$

Here $\Phi(q_2'^2, c_2)$ denotes the amplitude of reaction II, written down in terms of the momentum q_2' and the cosine of the scattering angle c_2 in its c.m.s. Im $\Phi(q_2'^2, c_2)$ is determined from (2.3) — (2.5). The symbol $\kappa_-(\delta, q_2'^2, c, q^2)$ denotes the value of (5.7) after the substitution (6.2).

7. DISCUSSION OF THE RESULTS

Relations (5.4), (5.6), (5.10), and (6.1) — (6.3) form a complete system of equations for the partial waves of πN scattering. In the derivation we took accurate account of the pole terms and of the contribution of the $\pi\pi$ interaction up to $q'^2 = -2.3\,\mu^2$, where the four-meson contributions to the process $\pi\pi \rightarrow N\overline{N}$ and the contributions due to the Mandelstam double representations of the spectral functions become significant. The equations obtained can therefore give sensible approximations only in the region of small energies, where we can confine ourselves to a small number of partial waves.

The equations contain the phase shifts δ_0 and δ_1 for the $\pi\pi$ scattering. Since we have at present no direct information on these phase shifts, it is natural to use the obtained system of equations as an indirect source of information on these phase shifts. Of great interest here is the verification of the hypothesis[16] that resonance exists in the $\pi\pi$-scattering p wave. Such a hypothesis leads to sensible results with respect to electromagnetic structure of the nucleon.[17]

We note that the procedure outlined for eliminating the kinematic singularities can be used from any other problems in the scattering of particles with unequal masses, for example, for Kπ and KN scattering.

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APPENDIX*

Let us explain the geometrical meaning of the transformation (4.6). We describe the kinematics of the reactions in a Lobachevskiĭ-type velocity space.[18] The usual connection between the elements of the triangle $C^2 = A^2 + B^2 - 2AB\cos\gamma$ is replaced here by $\cosh C = \cosh A \cosh B - \sinh A \sinh B \cos\gamma$. The 4-velocity p/m of the particle is represented by a point in this space.

We consider a plane in velocity space, passing through the velocities of the particles that participate in reaction I: $P_1 = p_1/M$, $Q_1 = q_1/\mu$ $(i = 1, 2)$ (see Fig. 3).

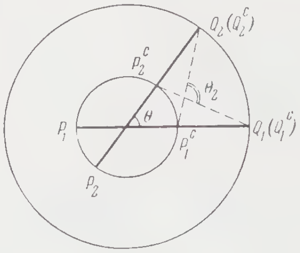

FIG. 3

The point 0 (intersection of the lines p_1Q_1 and p_2Q_2) represents the velocity of the center of mass. The invariants s and t can be expressed here in terms of the distances $\overline{P_1Q_1}$ and $\overline{Q_1Q_2}$:

$$s = M^2 + \mu^2 + 2M\mu \cosh \overline{P_1Q_1}, \quad t = 2\mu^2(1 - \cosh \overline{Q_1Q_2}).$$

In order to go over to the variable of the second reaction, we note that it follows from (3.2) and (4.5) that $ss^* = (M^2 - \mu^2)^2$. It follows therefore that s^* can be represented in the form

$$s^* = M^2 + \mu^2 - 2M\mu \cosh \overline{P_1^cQ_1}.$$

If we now identify P_1^C with the velocity of the initial nucleon in reaction II, and Q_1 with the velocity of the escaping meson in the reaction II $-Q_1^C$, then it follows from the invariance of t that the point Q_2 can be identified with the velocity Q_2^C. Then P_2^C is the intersection of the segment $0Q_2$ with the small circle. The velocity 0′ of the center of mass of the reaction II and the scattering angle θ_2 are determined by the intersection of the segment $P_2^CQ_1$ and $Q_2P_1^C$ and the angle between them. The momentum of reaction II is determined by the relation

$$q_2 = M \sinh \overline{O'P_1^c} = \mu \sinh \overline{O'Q_1^c}.$$

It is seen from the figure that $\theta_2 > \theta$, with the exception of the forward and backward scattering, when these two angles are equal, while $q_2 = q$ for backward scattering and is less than q for all other angles.

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*The interpretation given here was proposed by N. A. Chernikov.

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A METHOD OF SOLUTION OF FIELD-THEORETICAL PROBLEMS INVOLVING A STATIC NUCLEON

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A new method of solving field-theoretical problems involving a static nucleon is proposed. The formalism is not connected with the magnitude of the coupling constant and is based on the matrix methods for solving linear differential equations developed by Lappo-Danilevskiĭ. The solution is obtained in the form of a series for which the concrete form of the n-th term is known. The S matrix has been derived for a "charged" scalar theory with a static source. The renormalization constants are calculated by the method proposed. In this model, the transition to a point interaction does not lead to the appearance of logarithmic singularities in the renormalized charge g_r.

INTRODUCTION

THE assumption of weak coupling and the application of perturbation theory to the equations of mesodynamics lead to results which are not in agreement with experiment. It is therefore desirable to develop a method in which the coupling constant is not used as an iteration parameter and in which the approximations are based on a different principle. In this respect the Tamm-Dancoff method turned out to be useless on account of the renormalization difficulties. The method of dispersion relations has been advanced quite successfully in recent times; being based, however, on the most general principles of covariance, causality, unitarity, and the spectral hypothesis, this method contains less information than is involved in the specification of a Hamiltonian for the interacting fields. In view of the fact that the solution of the equations of quantum field theory is connected with great mathematical difficulties, various theoretical models have gained their well-known popularity.

The class of models with a "static source" attracts particular attention. In these models the fermion field is characterized only by the degrees of freedom of the spin and the isotopic spin. The circumstance that the experimental data on the interaction of π mesons and nucleons at low energies can be explained by the Chew-Low model,[1] which belongs to this class, indicates that this type of model does describe the actual interaction to some extent. It should be expected, therefore, that a number of problems of field theory are preserved under these simplifying assumptions. The knowledge of the exact solutions of such models may then be helpful for the understanding of the origins of the difficulties in the theory. However, even for the aforementioned class of models (with the exception of the trivial case of the interaction of neutral scalar mesons with a nucleon at rest[2]) there exist no methods of solution different from those enumerated above.

In the present paper we propose a new method of solving the equations of mesodynamics for this class of models on the example of a system of charged scalar mesons interacting with a static source.* Our formalism is not connected with the magnitude of the coupling constant; it is based on the matrix methods for solving linear differential equations developed by Lappo-Danilevskiĭ.[4] In the conventional language, the new formalism is equivalent to perturbation theory if one chooses the Hamiltonian of a system of neutral mesons and a static nucleon as the unperturbed Hamiltonian. However, the advantage is here that the n-th approximation term is written down in closed form, whereas in perturbation theory we can only find an arbitrary given term of the series, but not the general, n-th term. This circumstance allows us in principle to investigate the divergence of the series.

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*For an application of this method to a scalar symmetric theory with a static source, see reference 3.

1. REPRESENTATION OF THE S MATRIX IN THE FORM OF A CONTINUOUS INTEGRAL

Let us consider a system of charged scalar mesons interacting with a static "spread-out" nucleon. In this model the nucleon has only two isotopic degrees of freedom (proton and neutron). The system is described by the Hamiltonian

$$H = m_0(\psi^+\psi) + \frac{1}{2}\sum_{i=1}^{2}\int d\mathbf{x} : [\pi_i^2(\mathbf{x}) + (\nabla\varphi_i(\mathbf{x}))^2 + \mu^2\varphi_i^2(\mathbf{x})] :$$

$$+ g\sum_{i=1}^{2}\int d\mathbf{x}\,(\psi^+\tau_i\psi)\,\varphi_i(\mathbf{x})\,\rho(\mathbf{x}), \tag{1.1}$$

where $\pi_i(\mathbf{x})$ and $\varphi_i(\mathbf{x})$ are the meson field operators, $\psi = v_p c_p + v_n c_n$ is the operator of the nucleon field, c_N $(N = p, n)$ is the annihilation operator for the nucleon, v_N is the spinor describing the nucleon.

$$v_p = \begin{pmatrix}1\\0\end{pmatrix}, \qquad v_n = \begin{pmatrix}0\\1\end{pmatrix},$$

τ_i are the matrices for isotopic spin $1/2$, and

$$\rho(\mathbf{x}) = \sum_{\mathbf{k}} v(k)\,e^{i\mathbf{k}\mathbf{x}}$$

is the form factor of the nucleon.

In the interaction representation the S matrix satisfies the following equation:

$$i\partial S(t, t_0)/\partial t = H_I(t)\,S(t, t_0), \qquad S(t, t_0)|_{t=t_0} = 1;$$

$$H_I(t) = g\sum_{i=1}^{2}(\psi^+(t)\,\tau_i\psi(t))\,\hat{\varphi}_i(t), \qquad \psi(t) = \psi e^{-im_0 t},$$

$$\hat{\varphi}_i(t) = \int d\mathbf{x}\,\varphi_i(\mathbf{x})\,\rho(\mathbf{x}) = \sum_k \frac{v(k)}{\sqrt{2\omega_k}}[a_{ik}e^{-i\omega_k t} + a_{ik}^+ e^{i\omega_k t}]. \tag{1.2}$$

The solution of Eq. (1.2) can be written in the symbolic form

$$S(t, t_0) = T_\psi T_\varphi \exp\left\{-i\int_{t_0}^{t} d\xi\,H_I(\xi)\right\}. \tag{1.3}$$

The basic problem of the theory — the representation of the S matrix in terms of normal products — can be partially solved in a general form.[5] Indeed, it is possible to write the expression for the S matrix in a form that is ordered with respect to the meson operators $\hat{\varphi}$. The nucleon operators ψ and ψ^+, however, remain unordered (i.e., they remain under the sign of the T product). This partial ordering is achieved by representing the S matrix in the form of a continuous integral.

Following Feynman[5] and Bogolyubov and Shirkov,[6] one can write the S matrix, expressed as a normal product in the meson operators, in the form

$$S(t, t_0) = \iint \delta\Phi_1\,\delta\Phi_2 \exp\left\{-\frac{i}{2}\int_{t_0}^{t}\int_{t_0}^{t} d\xi\,d\eta\,\Phi_i(\xi)\,\Delta(\xi-\eta)\,\Phi_i(\eta)\right\} :$$

$$\times\exp\left\{i\int_{t_0}^{t} ds\,\hat{\varphi}_i(s)\,\Phi_i(s)\right\} : C^2\iint\delta\Lambda_1\,\delta\Lambda_2\exp\left\{-i\int_{t_0}^{t} ds\Lambda_i\right.$$

$$\left.\times(s)\,\Phi_i(s)\right\}\widetilde{S}(t, t_0\,|\,\Lambda_1\Lambda_2), \tag{1.4}$$

where the causal function $\Delta(\xi - \eta)$ is defined by

$$\langle 0\,|\,T\{\hat{\varphi}_i(\xi)\,\hat{\varphi}_j(\eta)\}\,|\,0\rangle = i\delta_{ij}\,\Delta(\xi-\eta)$$

$$= i\delta_{ij}\sum_{\mathbf{k}}\frac{v^2(k)}{2i\omega_k}\exp\{-i\omega_k|\xi-\eta|\}. \tag{1.5}$$

The functional integration in (1.4) goes over the space of real scalar functions Φ_i, Λ_i; C is a normalization constant.

The operator $\widetilde{S}(t, t_0\,|\,\Lambda_1\Lambda_2)$ has the meaning of an S matrix for a system of a classical charged meson field $\Lambda_1(t)$, $\Lambda_2(t)$ and a quantized nucleon field $\psi(t)$. It satisfies the equation

$$i\frac{\partial}{\partial t}\widetilde{S}(t, t_0\,|\,\Lambda_1\Lambda_2) = g\sum_{i=1}^{2}(\psi^+\tau_i\psi)\,\Lambda_i(t)\,\widetilde{S}(t, t_0\,|\,\Lambda_1\Lambda_2),$$

$$\widetilde{S}(t, t_0\,|\,\Lambda_1\Lambda_2)|_{t=t_0} = 1. \tag{1.6}$$

The problem of finding the S matrix to the Hamiltonian (1.1) is therefore divided into two parts: first, find the classical $\widetilde{S}$ matrix as a solution of Eq. (1.6) with arbitrary coefficient functions $\Lambda_1(t)$ and $\Lambda_2(t)$ and second, perform the functional integration over this matrix according to (1.4).

2. DETERMINATION OF THE "CLASSICAL" $\widetilde{S}$ MATRIX

Since the nucleon field has only two degrees of freedom and the operators of this field anticommute among themselves, the operator $\widetilde{S}(t, t_0\,|\,\Lambda_1\Lambda_2)$ can be expressed in the form of the following expansion in terms of the nucleon operators ψ and ψ^+ (which, as can be easily shown, is the most general one):

$$\widetilde{S}(t, t_0\,|\,\Lambda_1\Lambda_2) = 1 + [2(\psi^+\psi) - (\psi^+\psi)^2]\,f(t, t_0\,|\,\Lambda_1\Lambda_2)$$

$$+\sum_{i=1}^{3}(\psi^+\tau_i\psi)\,h_i(t, t_0\,|\,\Lambda_1\Lambda_2), \tag{2.1}$$

where f and h_i are ordinary scalar functions. This follows immediately from the relations

$$(\psi^+\tau_i\psi)(\psi^+\tau_j\psi) = i\varepsilon_{ijl}(\psi^+\tau_l\psi) + \delta_{ij}[2(\psi^+\psi) - (\psi^+\psi)^2],$$

$$(\psi^+\tau_i\psi)[2(\psi^+\psi) - (\psi^+\psi)^2] = (\psi^+\tau_i\psi)$$

which can be easily proved.

Substituting (2.1) in Eq. (1.9) and equating the coefficients of terms of identical structure, we obtain a system of equations for f and h_i, which can be written in matrix form:

$$i\partial Y(t, t_0 \mid \Lambda_1\Lambda_2)/\partial t = [\tau_1\Lambda_1(t) + \tau_2\Lambda_2(t)]\, Y(t, t_0 \mid \Lambda_1\Lambda_2),$$

$$Y(t, t_0 \mid \Lambda_1\Lambda_2)\big|_{t=t_0} = \mathrm{I},$$

$$Y(t, t_0 \mid \Lambda_1\Lambda_2) = \begin{pmatrix} 1 + f + h_3 & h_1 - ih_2 \\ h_1 + ih_2 & 1 + f - h_3 \end{pmatrix}. \qquad (2.2)$$

The solution of Eq. (2.2) presents considerable difficulties, since it reduces to the solution of a linear differential equation of second order with two arbitrary functions. Ordinarily equations of this type are solved by perturbation-theoretical methods, by expanding in terms of a parameter g which is assumed to be small. If the parameter g is large, on the other hand, Eq. (2.2) can be solved approximately with the help of the "quasi-classical" method. However, in the last case one obtains expressions which are not amenable to functional integration.

Lappo-Danilevskiĭ has developed a method of solving a system of differential equations using the theory of matrix functions. In this method the expansion parameter is not the constant g, but certain invariants of matrices appearing in the equation. We shall not dwell on the procedures to obtain the solution, but instead refer the reader to the extensive monograph of Lappo-Danilevskiĭ.[4] Omitting the long and unwieldy transformations of the recurrence relations of Lappo-Danilevskiĭ for the Eq. (2.2), we give at once the final expression for the solution:

$$Y(t, t_0 \mid \Lambda_1\Lambda_2)$$

$$= \sum_{q=0}^{\infty} \Bigg\{ \frac{(ig)^{2q}}{(2q)!} \int_{t_0}^{t} d\xi_1 \dots \int_{t_0}^{t} d\xi_{2q} \prod_{i=1}^{2q} \Lambda_1(\xi_i) \Bigg[\cosh\Bigg(ig \int_{t_0}^{t} ds\, \Lambda_2(s)$$

$$\times \prod_{j=1}^{2q} \varepsilon(s - \xi_j) \Bigg) - \tau_2 \sinh\Bigg(ig \int_{t_0}^{t} ds\, \Lambda_2(s) \prod_{j=1}^{2q} \varepsilon(s - \xi_j) \Bigg) \Bigg]$$

$$- \frac{(ig)^{2q+1}}{(2q+1)!} \int_{t_0}^{t} d\xi_1 \dots \int_{t_0}^{t} d\xi_{2q+1} \prod_{i=1}^{2q+1} \Lambda_1(\xi_i) \Bigg[\tau_1 \cosh\Bigg(ig$$

$$\times \int_{t_0}^{t} ds\, \Lambda_2(s) \prod_{i=1}^{2q+1} \varepsilon(s - \xi_j) \Bigg)$$

$$+ i\tau_3 \sinh\Bigg(ig \int_{t_0}^{t} ds\, \Lambda_2(s) \prod_{j=1}^{2q+1} \varepsilon(s - \xi_i) \Bigg) \Bigg] \Bigg\}, \qquad (2.3)$$

where $\epsilon(x) = 1$ for $x > 0$ and $\epsilon(x) = -1$ for $x < 0$. The functions $\Lambda_1(s)$ and $\Lambda_2(s)$ appear in (2.3) in a completely symmetric form, by expanding the hyperbolic cosine and sine in series and interchanging the order of summation we can obtain an expression for $Y(t, t_0 \mid \Lambda_1\ \Lambda_2)$ in which $\Lambda_1(s)$, τ_1 and $\Lambda_2(s)$, τ_2 change places.

It can be seen by direct substitution that the solution (2.3) satisfies Eq. (2.2) with the required initial conditions.

It is easy to write down a maximizing functional for the series (2.3), since the cosine and sine do not exceed the value unity (Λ_1 and Λ_2 are real) and the remaining series are easily summed:

$$Y(t, t_0) \mid \Lambda_1\Lambda_2) \leqslant (1 + \tau_1) \min \Bigg\{ \exp\Bigg[g \int_{t_0}^{t} ds\, |\Lambda_1(s)| \Bigg], \qquad (2.4)$$

$$\times \exp\Bigg[g \int_{t_0}^{t} ds\, |\Lambda_2(s)| \Bigg] \Bigg\}.$$

The solution of Eq. (2.2) is therefore given by the series (2.3) and a series which is obtained from the former by interchanging Λ_1 and Λ_2, and τ_1 and τ_2. These series converge uniformly and absolutely in the interval $[t, t_0]$, if at least one of the integrals $\int_{t_0}^{t} ds |\Lambda_1(s)|$ or $\int_{t_0}^{t} ds |\Lambda_2(s)|$ is bounded in $[t, t_0]$. (For the connection between the Lappo-Danilevskiĭ method and perturbation theory for equations of the type (2.2), see reference 3, Appendix A.

If $Y(t, t_0 \mid \Lambda_1\ \Lambda_2)$ is known, it is easy to write down the expression for the "classical" $\widetilde{S}$ matrix given by Eq. (2.1):

$$\widetilde{S}(t, t_0 \mid \Lambda_1\Lambda_2) = 1 - [2(\psi^+\psi) - (\psi^+\psi)^2]$$

$$+ \sum^{\infty} \Bigg\{ \frac{(ig)^{2q}}{(2q)!} \int_{t_0}^{t} d\xi_1 \dots \int_{t_0}^{t} d\xi_{2q} \prod_{i=1}^{2q} \Lambda_1(\xi_i) \Bigg[(2(\psi^+\psi)$$

$$- (\psi^+\psi)^2) \cosh\Bigg(ig \int_{t_0}^{t} ds\, \Lambda_2(s) \prod_{j=1}^{2q} \varepsilon(s - \xi_j) \Bigg)$$

$$- (\psi^+\tau_2\psi) \sinh\Bigg(ig \int_{t_0}^{t} ds\, \Lambda_2(s) \prod_{j=1}^{2q} \varepsilon(s - \xi_j) \Bigg) \Bigg]$$

$$- \frac{(ig)^{2q+1}}{(2q+1)!} \int_{t_0}^{t} d\xi_1 \dots \int_{t_0}^{t} d\xi_{2q+1} \prod_{i=1}^{2q+1} \Lambda_1(\xi_i) \Bigg[(\psi^+\tau_1\psi)$$

$$\times \cosh\Bigg(ig \int_{t_0}^{t} ds\, \Lambda_2(s) \prod_{j=1}^{2q+1} \varepsilon(s - \xi_j) \Bigg)$$

$$+ (\psi^+\tau_3\psi) \sinh\Bigg(ig \int_{t_0}^{t} ds\, \Lambda_2(s) \prod_{j=1}^{2q+1} \varepsilon(s - \xi_j) \Bigg) \Bigg] \Bigg\}. \qquad (2.5)$$

This formula is symmetric with respect to the interchange of the indices 1 and 2.

We note that the foregoing criterion for the uniform and absolute convergence is not sufficient for carrying out the functional integration, since the integration will always include functions Λ_1 and Λ_2 which do not conform to this criterion. However, we shall leave aside the question of the correct procedure for the functional integration, particularly since the existence of functional integrals has so far been shown only for a very narrow class of functionals. Let us assume that the series can be integrated term by term. This unjustified operation can be vindicated by the circumstance that the S matrix obtained as a result of the integration satisfies the basic Eq. (1.2), as can be seen by direct substitution.

3. DETERMINATION OF THE QUANTUM-MECHANICAL S MATRIX

The functional integration of the "classical" S matrix can be carried out without difficulty, since the solution of the classical equation has a Gaussian form. The method of calculating such functional integrals has been known since the work of Wiener;[7] its application to quantum-mechanical problems was developed by Feynman.[5] We only give the final expression for the S matrix (see reference 3):

$$
\begin{aligned}
S(t, t_0) = {} & 1 - [2(\psi^+\psi) - (\psi^+\psi)^2] \\
& + \sum_{q=0}^{\infty} \sum_{m=0}^{q} \Big\{ \frac{(ig)^{2q} i^m}{(2q-2m)!\, 2^m\, m!} \int_{t_0}^{t} d\xi_1 \ldots \\
& \times \int_{t_0}^{t} d\xi_{2q} \prod_{j=2}^{2m} \Delta(\xi_{j-1} - \xi_j) : \prod_{i=2m+1}^{2q} \hat{\varphi}_1(\xi_i) : \\
& \times \Big[(2(\psi^+\psi) - (\psi^+\psi)^2) : \cosh\Big(ig \int_{t_0}^{t} ds\, \hat{\varphi}_2(s) \prod_{k=1}^{2q} \varepsilon(s - \xi_k) \Big) : \\
& - (\psi^+\tau_2\psi) : \sinh\Big(ig \int_{t_0}^{t} ds\, \hat{\varphi}_2(s) \prod_{k=1}^{2q} \varepsilon(s - \xi_k) \Big) : \Big] \\
& \times \exp\Big(-\frac{ig^2}{2} \int_{t_0}^{t} ds_1 \int_{t_0}^{t} ds_2 \prod_{i=1}^{2q} \varepsilon(s_1 - \xi_i)\, \Delta(s_1 - s_2)\, \varepsilon(s_2 - \xi_i) \Big) \\
& - \frac{(ig)^{2q+1} i^m}{(2q+1-2m)!\, 2^m\, m!} \int_{t_0}^{t} d\xi_1 \ldots \int_{t_0}^{t} d\xi_{2q+1} \prod_{j=2}^{2m} \Delta(\xi_{j-1} - \xi_j) : \\
& \times \prod_{i=2m+1}^{2q+1} \hat{\varphi}_1(\xi_i) : \times \Big[(\psi^+\tau_1\psi) : \cosh\Big(ig \int_{t_0}^{t} ds\, \hat{\varphi}_2(s) \\
& \times \prod_{k=1}^{2q+1} \varepsilon(s - \xi_k) \Big) : + i(\psi^+\tau_3\psi) : \\
& \times \sinh\Big(ig \int_{t_0}^{t} ds \prod_{k=1}^{2q+1} \varepsilon(s - \xi_k)\, \hat{\varphi}_2(s) \Big) : \Big] \\
& \times \exp\Big(-\frac{ig^2}{2} \int_{t_0}^{t} ds_1 \int_{t_0}^{t} ds_2 \prod_{i=1}^{2q+1} \varepsilon(s_1 - \xi_i)\, \Delta(s_1 - s_2)\, \varepsilon(s_2 - \xi_i) \Big) \Big\}.
\end{aligned}
\tag{3.1}
$$

Expression (3.1) is symmetric with respect to interchanges of the indices 1 and 2; this corresponds to the symmetry of the classical function $Y(t, t_0 \mid \Lambda_1 \Lambda_2)$ reflected in the form of expression (2.3).

Our expression for the S matrix to the Hamiltonian (1.1) is written in a form which is normal in the nucleon as well as the meson operators. One sees by direct substitution that the S matrix satisfies Eq. (1.2) with the required initial conditions.

Thus the operation of functional integration, which is without rigorous mathematical foundation, leads in this case to the correct result, as can be verified by direct substitution.

By expanding in terms of the constant g we obtain the usual series of perturbation theory; however, in this case we have the advantage that we obtain the explicit form of the n-th term of this series, whereas in the present state of perturbation theory we can only obtain an ordinary given term of this series, but not the general n-th term. This defect of perturbation theory is, in our opinion, the main difficulty in investigating the convergence of the perturbation series.

To determine the physical meaning of the iterations in the S matrix (3.1) we turn again to Eq. (1.2). We have

$$
i\partial S(t, t_0)/\partial t = g\,[(\psi^+\tau_1\psi)\,\hat{\varphi}_1(t) + (\psi^+\tau_2\psi)\,\varphi_2(t)]\, S(t, t_0).
\tag{3.2}
$$

The charged meson operators are represented by the expressions $\hat{\varphi}_1 \pm \hat{\varphi}_2$, while the operators $\hat{\varphi}_1$ and $\hat{\varphi}_2$ create or annihilate a definite combination of positive and negative mesons; for example the operator $\hat{\varphi}_1$ corresponds to the combination $(\pi^- + \pi^+)/2$. Instead of the basic nucleon states v_p and v_n we introduce $v_+ = (v_p + v_n)/\sqrt{2}$ and $v_- = (v_p - v_n)/\sqrt{2}$. This transformation implies a transition to new vectors in the isotopic space, in terms of which the Eq. (3.2) is written as

$$
\begin{aligned}
i\partial S(t, t_0)/\partial t = {} & g\,[(\psi'^+\tau_3\psi')\,\hat{\varphi}_1(t) \\
& + (\psi'^+\tau_2\psi')\,\hat{\varphi}_2(t)]\, S(t, t_0).
\end{aligned}
\tag{3.3}
$$

Here $\psi' = v_+ c_+ + v_- c_-$; $c_\pm$ is the annihilation operator for the particles $v_\pm$.

The operator $\hat{\varphi}'$ enters Eq. (3.3) together with the diagonal matrix τ_3 and corresponds, therefore, to the emission and absorption of a combination of negative and positive mesons which does not cause transitions of the nucleon from the state v_+ to v_-, and vice versa. If the right-hand side of Eq. (3.3) did not contain a second term, we would have a neutral theory, in which the absorption and emission of a meson does not cause any changes of the isotopic coordinates of the nucleon. Solution (3.1) is equivalent to the perturbation theoretical solution, if the perturbation is given by the expression $(\psi'^+\tau_2\,\psi')\,\hat{\varphi}_2(t)$, which gives rise to transitions between the states v_+ and v_-.

We note that the matrix τ_2 standing together with the operator $\hat{\varphi}_2$ can be diagonalized by a different rotation in isotopic space: $v_\pm$ $(v_p \pm iv_n)/\sqrt{2}$. Then the "perturbation" term will be $(\psi'^{+}\tau_2\psi')\hat{\varphi}_1(t)$. This situation corresponds to the aforementioned symmetry of the S matrix with respect to the operators $\hat{\varphi}_1$ and $\hat{\varphi}_2$. However, if we restrict ourselves to a finite number of terms in the series (3.1), this symmetry will be destroyed (one operator appears in the exponent of an exponential and will turn up in all powers if the exponential is expanded, whereas the other operator will appear with a finite power).

Working with a cut-off series for the S matrix, we may encounter processes which violate the law of charge conservation. This occurs if we restrict ourselves to n terms of the series for processes which involve more than 2n mesons. It is therefore necessary to compute the matrix element from the complete series for the S matrix; only in the series for the matrix element can we restrict ourselves to a finite number of terms. Speaking in the language of perturbation theory, we can say that the separate terms of the series (3.1) include also graphs for which the law of charge conservation is not fulfilled, corresponding, for example, to the process $n \rightarrow p + \pi^{+}$. The complete S matrix satisfies the law of charge conservation exactly, and if the matrix elements are calculated in the correct manner, this law will not be violated, as was pointed out earlier. In our formalism we can therefore speak of a violation of the law of charge conservation in the virtual processes in analogy to the violation of the law of conservation of energy in virtual processes in the non-covariant formulation of perturbation theory.

4. RENORMALIZATION CONSTANTS

For the determination of the eigenfunctions and the eigenvalues of the Hamiltonian* (1.1) we use the adiabatic hypothesis for the inclusion of the interaction,[9] which can be formulated in the following fashion.

Let Φ_n be an eigenfunction of the free Hamiltonian H_0. If further the solution to the equation for the S^α matrix with adiabatically increasing interaction,

$$i\partial S^\alpha(t, t_0)/\partial t = H_I(t)e^{-\alpha|t|}S^\alpha(t, t_0),$$
$$S^\alpha(t, t_0)_{t=t_0} = 1, \tag{4.1}$$

is known, then the eigenfunctions of the operator $H = H_0 + H_I$ will be

$$C_n\Psi_n^{(\pm)} = \lim_{\alpha\to 0}[S^\alpha(0, \pm\infty)\Phi_n/(\Phi_n, S^\alpha(0, \pm\infty)\Phi_n)], \tag{4.2}$$

where C_n is a normalization constant and the signs ± correspond to "outgoing" and "incoming" waves, respectively.

The eigenvalue of the energy for the state $\Psi_n^{(\pm)}$ is determined by the equation

$$E_n = \lim_{\alpha\to 0}[(\Phi_n, HS^\alpha(0, \pm\infty)\Phi_n)/(\Phi_n, S^\alpha(0, \pm\infty)\Phi_n)]. \tag{4.3}$$

This quotient can be correctly defined by this limiting process, whereas the numerator and the denominator are indeterminate due to the presence of a phase factor.

The "adiabatic" S^α matrix, which is the solution of Eq. (4.1), can be easily obtained from (3.1) by replacing all differentials $d\xi_j$ by the expression $d\xi_j\exp(-\alpha|\xi_j|)$. We note that the introduction of compensation terms in the basic Hamiltonian leads automatically to the exclusion of infinite phases. Although the introduction of compensation terms is regarded as the more correct procedure, we find it more convenient to use the "adiabatic" theorems (4.2) and (4.3) in the calculation of the matrix elements.

Since the S matrix is given in the form of a series, the matrix elements will have the form of a limit for $\alpha \rightarrow 0$ of the ratio of two series. It turns out that, if one divides one series into the other and collects the terms corresponding to the same power of the coupling constant standing in front of the exponential under the integral, the phase of the terms obtained in this way cancels out, and we can therefore go to the limit $\alpha \rightarrow 0$ in each term separately. This procedure was illustrated in reference 3 on the example of the calculation of the renormalized coupling constant. The computations for the other matrix elements are analogous.

The resulting expressions are quite complicated. We therefore write down only the second and third approximations. The computation of the integrals is considerably simplified in the limiting case of a point interaction, where the form factor $v(k)$ tends to unity. Let us choose the following form for the form factor:

$$v(k) = \exp\{-(\omega_k - \mu)/2L\},$$

where L is interpreted as the largest cut-off momentum. The transition to a point interaction corresponds to letting L go to infinity.

Let us consider first of all the eigenvalue of the energy of the one-nucleon state. According to theorem (4.3) we find

---

*The Green's function to the Hamiltonian (1.1) was obtained in an earlier paper[8] by the authors.

$$E_N = \lim_{\alpha\to 0} \frac{\langle 0 | c_N H S^\alpha (0, -\infty) c_N^+ | 0\rangle}{\langle 0 | c_N S^\alpha (0, -\infty) c_N^+ | 0\rangle} = m_0 + \delta m, \tag{4.4}$$

$$\delta m = \lim_{\alpha\to 0} \frac{\langle 0 | c_N H_I S^\alpha (0, -\infty) c_N^+ | 0\rangle}{\langle 0 | c_N S^\alpha (0, -\infty) c_N^+ | 0\rangle} = \lim_{\alpha\to 0} \int_{-\infty}^{0} d\sigma e^{\alpha\sigma}\, 2g^2 \Delta(\sigma) \sum_{q=0}^{\infty} \frac{1}{q!} \left(\frac{-ig^2}{2}\right)^q A_q^\alpha \left[\sum_{q=0}^{\infty} \frac{1}{q!}\left(\frac{-ig^2}{2}\right)^q a_q^\alpha\right]^{-1} - g^2 \sum_k{}' \frac{v^2(k)}{\omega_k^2}$$
$$+ 2g^2 \int_{-\infty}^{0} d\sigma\, \Delta(\sigma) \exp\left\{-g^2 \sum_{\mathbf{k}} \frac{v^2(k)}{\omega_k^3}\left[1 - e^{i\omega_k\sigma}\right]\right\} + \dots . \tag{4.5}$$

Here

$$A_q^\alpha = \int_{-\infty}^{0} d\xi_1 \dots \int_{-\infty}^{0} d\xi_{2q} \exp\{\alpha(\xi_1 + \dots + \xi_{2q})\} \prod_{i=2}^{2q} \Delta(\xi_{i-1} - \xi_i) \prod_{j=1}^{2q} \varepsilon(\sigma - \xi_j)$$
$$\times \exp\left\{-\frac{ig^2}{2} \int_{-\infty}^{0} ds_1 \int_{-\infty}^{0} ds_2 e^{\alpha(s_1+s_2)} \prod_{k=1}^{2q} \varepsilon(s_1 - \xi_k)\, \Delta(s_1 - s_2)\, \varepsilon(s_2 - \xi_k)\right\},$$

$$a_q^\alpha = \int_{-\infty}^{0} d\xi_1 \dots \int_{-\infty}^{0} d\xi_{2q} \exp\{\alpha(\xi_1 + \dots + \xi_{2q})\} \prod_{i=2}^{2q} \Delta(\xi_{j-1} - \xi_j) \exp\left\{-\frac{ig^2}{2} \int_{-\infty}^{0} ds_1 \int_{-\infty}^{0} ds_2 \prod_{k=1}^{2q} \varepsilon(s_1 - \xi_k)\, \Delta(s_1 - s_2)\, \varepsilon(s_2 - \xi_k)\right\}.$$

In the limiting case of a point interaction the renormalization of the mass is given by

$$\delta m \to -\frac{g^2}{2\pi^2} L\left[1 + \frac{1}{2(g^2/\pi^2 + 1)} + \dots\right] \text{ for } L \to \infty. \tag{4.6}$$

The renormalization constant for the fermion field Z_2 is, in conformity with its interpretation as a probability, given by

$$Z_2 = |\langle 0 | c_N c^\alpha (0, -\infty)\, \overset{+}{c}_N | 0\rangle|^2 = \left|\sum_{q=0}^{\infty} \frac{1}{q!}\left(\frac{-ig^2}{2}\right)^q a_q^\alpha\right|^2$$
$$= \exp\left\{-g^2 \sum_{\mathbf{k}} \frac{v^2}{2\omega^3}\right\}\left[1 - g^2\, \mathrm{Re} \int_0^\infty d\eta \int_0^\infty d\nu\, \Delta(\eta)\right.$$
$$\left.\times \exp\left\{g^2 \sum_{\mathbf{k}} \frac{v^2}{\omega^3}\left[-1 + e^{-i\omega\eta} + e^{-i\omega\nu} - e^{-i\omega(\nu+\eta)}\right]\right\} + \dots\right]. \tag{4.7}$$

The series inside the square brackets contains an undetermined phase factor $e^{iA/\alpha}$, which drops out when the absolute square of this series is taken. Restricting ourselves to the first two terms, we find in the limit L

$$Z_2 = L^{-g^2/1\pi^2}\left[1 + \frac{g^2 \quad \pi^2}{g^2/\pi^2 + 1} \ln L + \dots\right]. \tag{4.8}$$

The most interesting physical consequence of our discussion is the connection between the renormalized (observed) coupling constant g_r and the bare coupling constant g. This connection is given by the ratio

$$\frac{g_r}{g} = \lim_{\alpha\to 0} \frac{\langle 0 | c_p S^\alpha (\infty, 0) (\psi^+ \tau_+ \psi) S^\alpha (0, -\infty) c_n^+ | 0\rangle}{\langle 0 | c_p S^\alpha (\infty, -\infty) c_p^+ | 0\rangle}. \tag{4.9}$$

To facilitate the following analysis we shall assume that the field $\hat{\varphi}_1$ enters in the interaction Hamiltonian (1.1) with the coupling constant g_1 and the field $\hat{\varphi}_2$ with the coupling constant g_2.

Restricting ourselves to the first few terms of the series, we find after some calculation (see reference 3)

$$\frac{g_r}{\sqrt{g_1 g_2}} = 1 + g_1^2 \int_0^\infty dx\, x \left(\sum_{\mathbf{k}} \frac{v^2}{\omega} e^{-i\omega x}\right) \exp\left\{-2g_2^2 \sum_{\mathbf{k}} \frac{v^2}{\omega^3}(1 - e^{-i\omega x})\right\}$$

$$- g_1^4 \int_0^\infty dx_1 \int_0^\infty dx_2 \int_0^\infty dx_3\, 2x_1 \left(\sum_{\mathbf{k}_1} \frac{v_1^2}{\omega_1} e^{-i\omega_1 x_1}\right)\left(\sum_{\mathbf{k}_2} \frac{v_2^2}{\omega_2} e^{-i\omega_2 x_2}\right)$$

$$\times \exp\left\{-2g_2^2 \sum_{\mathbf{k}} \frac{v^2}{\omega^3}(2 - e^{-i\omega x_1} - e^{-i\omega x_2})\right\}$$

$$\times \left[\exp\left\{-2g_2^2 \sum_{\mathbf{k}} \frac{v^2}{\omega^3}\left(e^{-i\omega(x_1+x_3)} + e^{-i\omega(x_2+x_3)}\right.\right.\right.$$

$$\left.\left.\left. - e^{-i\omega(x_1+x_2+x_3)} - e^{-i\omega x_3}\right)\right\} - 1\right]$$

$$- g_1^4 \int_0^\infty dx_1 \int_0^\infty dx_2 \int_0^\infty dx_3\, 2x_1 \left[\left(\sum_{\mathbf{k}_1} \frac{v_1^2}{\omega_1} e^{-i\omega_1(x_1+x_3)}\right)\right.$$

$$\times \left(\sum_{\mathbf{k}_2} \frac{v_2^2}{\omega_2} e^{-i\omega_2(x_2+x_3)}\right) + \left(\sum_{\mathbf{k}_1} \frac{v_1^2}{\omega_1} e^{-i\omega(x_1+x_2+x_3)}\right)\left(\sum_{\mathbf{k}_2} \frac{v_2^2}{\omega_2} e^{-i\omega_2 x_3}\right)$$

$$\times \exp\left\{-2g_2^2 \sum_{\mathbf{k}} \frac{v^2}{\omega^3}\left(2 - e^{-i\omega x_1} - e^{-i\omega x_2} + e^{-i\omega(x_1+x_3)}\right.\right.$$

$$\left.\left.\left. + e^{-i\omega(x_2+x_3)} - e^{-i\omega(x_1+x_2+x_3)} - e^{-i\omega x_3}\right)\right\}\right] + \dots . \tag{4.10}$$

We note that the constants g_1 and g_2 can be interchanged; this is a consequence of the symmetry of the S matrix with respect to the operators $\hat{\varphi}_1$ and $\hat{\varphi}_2$, as has already been emphasized several times.

Formula (4.10) is remarkable in that it has a finite limit if the cut-off is removed, i.e., if L → ∞ (see reference 3). Setting $g_1 = g_2 = g$, we have

$$\frac{g_r}{g} = 1 - \frac{1}{g^2/\pi^2 + 1} - \frac{g^4}{2\pi^4}\int_0^\infty dx_1 \int_0^\infty dx_2 \frac{x_1 + x_2}{(1 + x_1)^{2+g^2/\pi^2}(1 + x_2)^{2+g^2/\pi^2}}$$

$$\times \int_0^\infty dx_3 \left[\left(1 + \frac{x_1 x_2}{(1 + x_3)(1 + x_1 + x_2 + x_3)}\right)^{g^2/\pi^2} - 1\right]$$

$$- \frac{g^4}{2\pi^4}\int_0^\infty dx_1 \int_0^\infty dx_2 \int_0^\infty dx_3 \frac{x_1 + x_2}{(1 + x_1)^{g^2/\pi^2}(1 + x_2)^{g^2/\pi^2}}\Bigg(1$$

$$+ \frac{x_1 x_2}{(1 + x_1 + x_2 + x_3)(1 + x_3)}\Bigg)^{g^2/\pi^2}\left[\frac{1}{(1 + x_1 + x_3)^2(1 + x_2 + x_3)^2}\right.$$

$$\left. - \frac{1}{(1 + x_1 + x_2 + x_3)^2(1 + x_3)^2}\right]. \qquad (4.11)$$

Let us consider the first term in expression (4.10) in more detail:

$$g_1^2 \int_0^\infty x dx \left(\sum_{\mathbf{k}} \frac{v^2}{\omega} e^{-i\omega x}\right) \exp\left\{- 2g_2^2 \sum_{\mathbf{k}} \frac{v^2}{\omega^3}(1 - e^{-i\omega x})\right\}. \qquad (4.12)$$

It is easily seen that we obtain a series containing terms which are logarithmically divergent in L, if we expand the function under the integral in terms of g_2^2. The most divergent part of this series has the form

$$g_1^2 \ln L \sum_{n=0}^{\infty} (- g_2^2 \ln L)^n / (n + 1)! \qquad (4.13)$$

in complete correspondence with the perturbation theoretical result. At the same time the expression (4.12) has the limit $-g_1^2 [g_2^2 (1 + g_2^2/\pi^2)]^{-1}$ for $L \to \infty$.

The integral (4.12), as a function of g_2^2, therefore has a pole at the point $g_2^2 = 0$ and can thus not be expanded into a Taylor series in the neighborhood of $g_2^2 = 0$. The situation is the same for the subsequent terms of the series. However, the convergence radii for the integrals change from one order to the next. Thus the third integral in (4.10) converges already for $g_2^2/\pi^2 > 1$, and in n-th order the integrals converge for $g_2^2/\pi^2 > (n - 1)$. It is therefore necessary that g_2^2/π^2 is assumed to be infinite in order that all terms of the series (4.9) be finite if the cut-off is removed ($L \to \infty$).

These restrictions on the constant g_2^2, which are different for each term of the series, seem to be rather meaningless. In order to find an explanation for this, we recall that the expression for the renormalized constant (4.9) is symmetric under the interchange $g_1 \rightleftarrows g_2$. Hence all conclusions concerning g_2 are also true for g_1 (since (4.9) can be expressed in the form of a series in g_1, with g_2 appearing only in the exponent of the exponential), i.e., it can be asserted that we also have a singularity for $g_1 = 0$. Thus $g_r = f(g_1, g_2)$ cannot be written as an expansion in the neighborhood of $g_1 = 0$ or $g_2 = 0$. But the series (4.11) is an expansion precisely about $g_1 = 0$. This is apparently the explanation for the senseless result which we have been talking about.

CONCLUSION

Our method for solving field theoretical problems involving a static nucleon leads to solutions in the form of a series for which the n-th term is known. The coupling constant is not an expansion parameter, so that we need not make any assumptions concerning its magnitude. It can be hoped that the knowledge of the explicit form of the n-th term of the series representing the solution will allow us in the future to decide the problem of the convergence of the series, at least for separate models of this class. However, the investigation of the renormalized coupling constant seems to indicate that there exist poles at the point $g = 0$ in the exact solution for some models. This result casts serious doubt on all methods using an expansion in the constant g. In any case, it follows from formula (4.2) that the logarithmically divergent terms, which are absent in our solution, inevitably turn up in the expansion in g. We further note the following: the fact that the application of the largely unexplored method of functional integration leads in our case to the correct results gives rise to the hope that this method will, after some further perfection, be more effectively used in the solution of the exact equations of field theory.

In conclusion we regard it as our pleasant duty to express our deep gratitude to Professor D. I. Blokhintsev and Academician N. N. Bogolyubov for very useful and stimulating comments on this work.

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SOVIET PHYSICS JETP VOLUME 12, NUMBER 2 FEBRUARY, 1961

PROBABILITIES OF ROTATIONAL GAMMA TRANSITIONS OF TYPE E2 AND QUADRUPOLE MOMENTS OF DEFORMED NUCLEI WITH K = 1 AND ½

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Expressions are given for B(E2) and Q for cases where these quantities depend not only on the direct matrix element, but also on the cross matrix element.

THE quadrupole moments Q of deformed nuclei and the reduced probabilities B for γ transitions of type E2 between levels of a rotational band may have anomalous values for the case where K = 1 or ½ (K is the projection of the total angular momentum on the nuclear symmetry axis). For K = 1 and ½ the quantities Q and B(E2) depend not only on the intrinsic quadrupole moment $Q_0 \equiv \langle \chi_K | \hat{Q} | \chi_K \rangle$ but also on the cross matrix element $\langle \chi_{-K} | \hat{Q} | \chi_K \rangle$ ($\hat{Q}$ is the quadrupole moment operator, χ_K is the function characterizing the internal state of the nucleus).

The anomaly in the values of Q and B(E2) in nuclei with K = 1 and ½ is similar to the well-known anomaly in the magnetic moments μ and reduced probabilities for rotational transitions, B(M1), in nuclei with K = ½, and is due to the equivalence of positive and negative directions of the nuclear axis.

The quadrupole moment of a nucleus with angular momentum I and projection K = 1 can be written as follows:

$$Q = Q_0 \frac{3 - I(I+1)\,[1 + (-1)^I\, 3b_0]}{(I+1)(2I+3)}. \qquad (1)$$

In particular, for the ground state of the rotational band where I = K = 1,

$$Q = Q_0 (1 + 6b_0) / 10. \qquad (2)$$

The coefficient b_0 characterizes the ratio of the matrix elements $\langle \chi_{-1} | \hat{Q} | \chi_1 \rangle / \langle \chi_1 | \hat{Q} | \chi_1 \rangle$. The reduced probabilities for E2 γ transitions between levels of a rotational band with K = 1 can be expressed in terms of the same parameters Q_0 and b_0. For transitions with I + 1 → I,

$$B(E2) = \frac{5}{16\pi} e^2 Q_0^2 (C^{IK}_{I+1K;\,20})^2 [1 - (-)^{I-K} (I + 1)\, b_0]^2, \qquad (3)$$

For transitions with I + 2 → I,

$$B(E2) = \frac{5}{16\pi} e^2 Q_0^2 (C^{IK}_{I+2K;\,20})^2 [1 + (-)^{I-K}\, b_0]^2, \qquad (4)$$

where $C^{\cdot\,\cdot}_{\cdot\,\cdot\,\cdot}$ are Clebsch-Gordan coefficients.

In the case of K = ½, the quadrupole moment of the ground state of the rotational band I = K = ½ is identically equal to zero. The expressions for the reduced probabilities B(E2) for transitions between levels of a rotational band with K = ½ have the same form as (3) and (4), and differ only in the values of I and K. The coefficient b_0 characterizes the ratio $\langle \chi_{-1/2} | Q | \chi_{1/2} \rangle / \langle \chi_{1/2} | Q | \chi_{1/2} \rangle$.

The magnitude and sign of the coefficient b_0 are determined by the internal state of the nucleus. In particular cases it may turn out that the cross matrix element is small compared to Q_0. However, there is no basis for assuming that $b_0 \ll 1$ for all nuclei. Therefore the measurement of a single value for B(E2) in nuclei with K = 1 and ½ is insufficient for determining the intrinsic quadrupole moment and the deformation parameter of the nucleus.

There are a considerable number of deformed nuclei known at present which have states with K = 1 or ½. But the corresponding experimental data are available only for five nuclei with K = ½. These data are given in the table. There we also give the theoretical values calculated on the assumption that $b_0 = 0$. Agreement of experi-

Nucleus	Ratio	Theory for $b_0 = 0$	Experiment
Yb^{171}	$\frac{B(E\,2,\,^1/_2 \to {}^5/_2)}{B(E\,2,\,^1/_2 \to {}^3/_2)}$	1.50	1.49 [1]
Tm^{169}	$\frac{B(E\,2,\,^5/_2 \to {}^3/_2)}{B(E\,2,\,^5/_2 \to {}^1/_2)}$	0.28	0.31 [2,3]
W^{183}	$\frac{B(E\,2,\,^5/_2 \to {}^3/_2)}{B(E\,2,\,^5/_2 \to {}^1/_2)}$	0.28	0.52 [4,5]
U^{235}	$\frac{B(E\,2,\,^5/_2 \to {}^3/_2)}{B(E\,2,\,^5/_2 \to {}^1/_2)}$	0.28	0.16 [6,7]
Pu^{239}	$\frac{B(E\,2,\,^5/_2 \to {}^3/_2)}{E(E\,2,\,^5/_2 \to {}^1/_2)}$	0.28	1.04 [8,9]

mental and theoretical values occurs only for Yb^{171} and Tm^{169}. For the other nuclei one observes deviations, i.e., b_0 and consequently also $\langle\chi_{-K}|\hat{Q}|\chi_K\rangle$ are different from zero. However, the precision of the available experimental data is low, so that accurate measurements of Q and B(E2) for nuclei with K = 1 and $\frac{1}{2}$ would have considerable interest.

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SOVIET PHYSICS JETP VOLUME 12, NUMBER 2 FEBRUARY, 1961

CONTRIBUTION TO THE THEORY OF SIMPLE MAGNETOHYDRODYNAMIC WAVES

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The Riemann invariants are computed for simple magnetohydrodynamic waves. The change of velocity in fast and slow magnetoacoustic waves is determined for the case when the magnetic pressure in front of the wave is much smaller than the hydrostatic pressure.

1. BASIC EQUATIONS

We use the name "simple waves" for those solutions of the equations of magnetohydrodynamics, which have the form of a traveling wave[1]

$$u_i = f_i\,(x \pm Vt) \qquad (i = 1,\ 2, \ldots, n), \tag{1}$$

where $u_1, u_2 \ldots, u_n$ is the totality of the magnetohydrodynamic quantities such as the pressure p, the density ρ, the three velocity components v_x, v_y, and v_z, and the two transverse components of the magnetic field H_y and H_z (we confine ourselves to plane one-dimensional waves, propagating in the x direction, so that H_x is constant). In the magnetohydrodynamics of an ideally conducting non-viscous medium, there are seven magnetohydrodynamic quantities n. It follows from (1.1) that in a simple wave all the magnetohydrodynamic quantities can be expressed in terms of one among them, for example ρ, which in turn depends on the coordinates and on the time.*

The quantity V, which enters in (1.1), is a function of ρ. This means that, in a simple wave, points which have different densities are displaced with different velocities. Thus, the profile of the simple wave is distorted as the wave propagates. It can be shown that the steepness of the wave decreases in the rarefaction regions ($d\rho/dt < 0$) and increases in the compression regions ($d\rho/dt > 0$).[4,6,7] This leads, in final analysis, to the formation of discontinuities (shock waves) in the compression regions.

The simple wave plays a special role in magnetohydrodynamics because in the absence of discontinuities, only a simple wave can have a boundary with a region of a steady flow.[3,5] It must be noted, however, that this theorem is somewhat of lesser importance in magnetohydrodynamics than in ordinary hydrodynamics. Thus, when a piston moves out in a non-conducting medium that obeys the equations of ordinary hydrodynamics, the region of steady flow along the characteristic $dx/dt = c$ (c is a velocity of sound) bounds on a simple wave that reaches the piston (see Fig. 1A). On the other hand, if a piston moves in a magnetohydrodynamic medium, two characteristics emerge from the point $x = 0$, $t = 0$: a 'fast' characteristic, $dx/dt = U_+$, and a 'slow' characteristic $dx/dt = U_-$ (U_+ and U_- are the propagation speeds of the fast and slow magnetoacoustic wave, see Fig. 1b). In the region enclosed between the two characteristics, a simple wave is produced, whereas in the region enclosed between the slow characteristic and the piston, the motion of the medium is in general not described by a simple wave.*

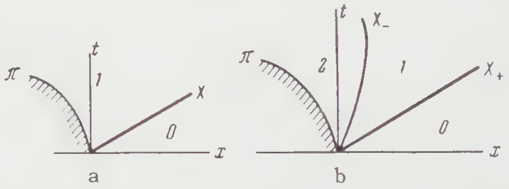

FIG. 1. Waves produced in the motion of the piston: a – in ordinary hydrodynamics, b – in magnetohydrodynamics. Π – line of motion of piston; X – characteristic in ordinary hydrodynamics, $dx/dt = v_x + c$; X_+ – 'fast' characteristic in magnetohydrodynamics, $dx/dt = v_x + U_+$; X_- – 'slow' characteristic in magnetohydrodynamics, $dx/dt = v_x + U_-$; 0 – standstill region; 1 – simple wave; 2 – non-simple wave.

However, if the piston moves with constant velocity, then the problem does not contain a parameter with the dimension of length. Therefore all the magnetohydrodynamic quantities depend on the coordinates and the time only through the combination x/t. Such waves, called self-similar, are a particular case of simple waves. Self-similar waves are always rarefaction waves.[6,7]

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*The first published paper on simple waves in magnetohydrodynamics is that of Owens.[2] The investigation of simple waves for $H_x \neq 0$ is the subject of papers by Friedrichs and Lax,[3] Kulikovskiĭ,[4] Akhiezer, Lyubarskiĭ, and Polovin.[5-7]

*This can be seen even from the fact that a simple wave is characterized by one arbitrary function,[5] whereas on the surface of the piston there should be satisfied boundary conditions[8] characterized by two arbitrary functions, $v_x(t)$ and $v_y(t)$.

In magnetohydrodynamics there are three types of simple waves:[5] fast and slow magnetoacoustic waves, Alfven waves, and entropy waves. The greatest interest attaches to magnetoacoustic waves. The latter are always plane: if the velocity and magnetic field vectors in the xy plane ($v_z = 0$, $H_z = 0$) at the initial instant then these relations will continue to be satisfied.

If the medium is described by the equation of state of an ideal gas, $p\rho^{-\gamma} = \text{const}$, then, as shown by Friedrichs (see reference 9), the integration of the equation of simple waves reduces to the integration of a linear differential equation

$$dr / dq_{\pm} = \theta (rq_{\pm}^2 - 1) / q_{\pm}^2 (q_{\pm} - 1),$$

$$r = c^2 / U_x^2 \equiv 4\pi\gamma p / H_x^2, \qquad \theta = \gamma / (2 - \gamma),$$

$$q_{\pm} = U_{\pm}^2 / c^2, U_{\pm} = \{[U^2 + c^2 \pm \sqrt{(U^2 + c^2)^2 - 4c^2 U_x^2}]/2\}^{1/2}, \tag{1.2}$$

where c is the velocity of sound, equal to $\gamma p/\rho$; U is the Alfven velocity, equal to $\mathbf{H}/\sqrt{4\pi\rho}$. As can be seen from the definition, r is a dimensionless pressure.

Equation (1.1) makes it possible to determine the quantity $q_{\pm}$ as a function of r, after which the velocity components are found from the equations

$$dv_x / dr = \varepsilon c \sqrt{q_{\pm}} / \gamma r, \tag{1.3}$$

$$dv_y / dr = \mp \varepsilon c \gamma^{-1} r^{-1} (q_{\pm} - 1)^{1/2} (rq_{\pm} - 1)^{1/2} \operatorname{sign} (H_x H_{1y}). \tag{1.4}$$

The upper and lower signs in (1.4) correspond to the fast and slow waves, respectively; $\epsilon = +1$ for waves propagating in the direction of positive x and $\epsilon = -1$ corresponds to the opposite direction. The subscript 1 (designating H_y) pertains to the region in front of the wave.

The transverse magnetic field H_y is determined from

$$H_y = H_x \sqrt{(q_{\pm} - 1)(rq_{\pm} - 1) / q_{\pm}} \operatorname{sign} H_{1y}. \tag{1.5}$$

2. QUALITATIVE INVESTIGATION

From the equations of simple waves it follows that the transverse magnetic field Hy decreases in a fast self-similar wave and increases in a slow wave.[6,7] If the wave propagates in the direction of positive x, then the longitudinal velocity component v_x decreases in both slow and fast magnetoacoustic waves. If, in addition, the quantity $H_x H_y$ is positive, then the transverse velocity component v_y increases in the fast magnetoacoustic wave and decreases in a slow one.

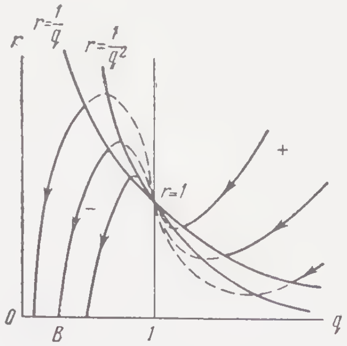

FIG. 2. Integral curves of Eq. (1.1). The + sign designates the fast waves and the − sign the slow waves. The non-existing portions of the integral curves (on which the transverse magnetic field H_y becomes imaginary) are shown dashed. The arrows indicate the direction of displacement in the plane (q, r) in the self-similar wave. The region of the abscissa axis on which a vacuum is produced behind the slow wave is designated by the letter B.

From the definition of the quantities r and $q_{\pm}$ it follows that the inequalities

$$rq_+ \geqslant 1, \quad q_+ > 1, \qquad rq_+^2 > 1, \tag{2.1}$$

are satisfied in a fast magnetoacoustic wave, while in a slow wave we have

$$rq_- \leqslant 1, \quad q_- < 1, \qquad rq_-^2 < 1. \tag{2.2}$$

From inequalities (2.1) and (2.2) and from Eq. (1.2) it follows that $dr/dq_{\pm} > 0$. Since the pressure r decreases in self-similar waves, the quantities $q_{\pm}$ also decrease (in both slow and fast magnetoacoustic waves). The first inequality in (2.1) or (2.2), namely $rq_{\pm} \geq 1$, determines the simple wave with maximum amplitude. Magnetoacoustic waves in which $rq_{\pm} = 1$ will be called complete. It is seen from relations (2.1) and (2.2) that the equality $rq_{\pm} = 1$ can be reached behind a fast wave or in front of a slow wave.

When $rq_{\pm} = 1$ the transverse magnetic field vanishes. The point $r = 1$; $q_{\pm} = 1$ is a singular point (node) of (1.2). In the vicinity of the singular point, the equation of the integral curves has the form

$$r = 1 - \gamma (q - 1) / (\gamma - 1) + C (q - 1)^{\theta},$$

where C is a constant of integration. At the singular point all the integral curves have a common tangent, with a slope

$$dr / dq = -\gamma / (\gamma - 1).$$

At the point of intersection of the integral curves with the line $rq^2 = 1$, the derivative dr/dq vanishes and the integral curves have horizontal tangents.

The overall shape of the integral curves of (1.2) is shown in Fig. 2. The non-existent portions of the integral curves (corresponding to imaginary values of Hy) are indicated by dashed lines. The arrows indicate the direction of displacement in the plane (r, q) upon propagation of a self-similar wave. The fast and slow magnetoacoustic waves are marked "+" and "−" respectively. On the line $r = 0$ the pressure vanishes — cavitation sets in. This line is indicated in Fig. 2 by the letter B. As can be seen from Fig. 2, cavitation can take place only in a slow magnetoacoustic wave. It is also seen from Fig. 2 that the inequality $r < 1$ is satisfied behind a complete fast wave, and the inequality $r > 1$ is satisfied ahead of a complete slow wave.*

3. THE RIEMANN INVARIANTS

In ordinary hydrodynamics, the Riemann invariant is defined as a function J of the hydrodynamics quantities $u_1, u_2, \ldots, u_n$, which remains constant along a certain characteristic. According to this definition, there are no Riemann invariants in magnetohydrodynamics. In fact, the equations of magnetohydrodynamics can be written schematically in the form

$$\frac{\partial u_i}{\partial t} + \sum_{k=1}^{n} X_{ik}(u_1, \ldots, u_n)\frac{\partial u_k}{\partial x} = 0 \qquad (i = 1, \ldots, n). \tag{3.1}$$

Since the Riemann invariant $J(u_1, \ldots, u_n)$ is constant along the characteristic, we have

$$\frac{\partial J}{\partial t} + V(u_1, \ldots, u_n)\frac{\partial J}{\partial x} = 0, \tag{3.2}$$

where V is the slope of the characteristic.

Equation (3.2) should be the consequence of (3.1). There should therefore exist functions μ_i $(u_1, \ldots, u_n)$ such that the equation

$$\sum_i \mu_i \frac{\partial u_i}{\partial t} + \sum_{i,k} \mu_i X_{ik}\frac{\partial u_k}{\partial x} = 0. \tag{3.3}$$

becomes identical with Eq. (3.2), which we write in the form

$$\sum_i \frac{\partial J}{\partial u_i}\frac{\partial u_i}{\partial t} + \sum_k V\frac{\partial J}{\partial u_k}\frac{\partial u_k}{\partial x} = 0. \tag{3.4}$$

Comparing (3.3) and (3.4), we find

$$\mu_i = \partial J / \partial u_i, \tag{3.5}$$

$$\sum \mu_i X_{ik} = V\partial J / \partial u_k = V\mu_k. \tag{3.6}$$

From relations (3.5) we obtain $n(n-1)/2$ equations

$$\partial\mu_i/\partial x_k = \partial\mu_k/\partial x_i,$$

which must be satisfied by the functions μ. The system (3.6) contains an additional $(n-1)$ equation. Thus, the n functions $\mu_1, \ldots, \mu_n$ must satisfy $(n-1) + n(n-1)2 = (n-1)(n+2)/2$ equations, which is generally impossible if $n > 2$.* Consequently, the definition given above for the Riemann invariants cannot be extended to include magnetohydrodynamics.

However, it is possible to modify the definition of the Riemann invariant so as to make it meaningful in magnetohydrodynamics.[3] We shall define as a Riemann invariant a function of the magnetohydrodynamic quantities $J(u_1, \ldots, u_n)$, which remains constant in a simple wave. Inasmuch as simple waves are described by a system of ordinary differential equations, there exist $(n-1)$ integrals of the system of equations, and these are the Riemann invariants. To each simple wave there correspond thus $(n-1)$ Riemann invariants.

A simple magnetoacoustic wave, according to (1.2) — (1.4), has four Riemann invariants:

$$J_1 = r(q-1)^{-\theta} + \theta\int q^{-2}(q-1)^{-\theta-1}dq, \tag{3.7}$$

$$J_2 = v_x + \varepsilon\gamma^{-1}\int c\sqrt{q}\,r^{-1}\,dr, \tag{3.8}$$

$$J_3 = v_y \pm \varepsilon\gamma^{-1}\operatorname{sign}(H_x H_{1y}) \cdot \int (q-1)^{1/2}(rq-1)^{-1/2}cr^{-1}\,dr, \tag{3.9}$$

$$J_4 = p\rho^{-\gamma}. \tag{3.10}$$

When $\gamma = 5/3$, the invariant J_1 is expressed in terms of elementary functions:

$$J_1 = (r-1)(q-1)^{-5} + \frac{5}{2}(q-1)^{-4} - 5(q-1)^{-3} + 10(q-1)^{-2} - 25(q-1)^{-1} - 5q^{-1} + 30\ln(q/q-1|). \tag{3.11}$$

The invariants J_2 and J_3 make it possible to determine the discontinuities in the quantities v_x and v_y:

$$\Delta_\pm v_x = -\varepsilon\gamma^{-1}\int_{r_2}^{r_1} c\sqrt{q}\,r^{-1}dr, \tag{3.12}$$

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*It follows therefore that the fast and slow waves cannot be simultaneously complete. A complete slow wave can exist only when there is no fast wave and when the transverse magnetic field H_y in front of the wave vanishes.

*Equations (3.5) and(3.6) become compatible through accidental degeneracy. It is easily shown, however, that there is no such degeneracy in magnetohydrodynamics, and equations (3.5) and (3.6) are incompatible.

$$\Delta_{\pm}v_y = \pm\varepsilon\gamma^{-1}\,\mathrm{sign}\,(H_xH_{1y})\cdot\int\limits_{r_2}^{r_1}(q-1)^{1/2}(rq-1)^{-1/2}cr^{-1}\,dr. \tag{3.13}$$

The index 2 pertains to the region behind the wave.

As was shown by Lax,[3] the Riemann invariants are conserved in shock waves of low intensity, accurately to quantities of second order inclusive.

4. FAST WAVE WHEN $U_1 \ll c_1$

Equations (3.12) and (3.13) become much simpler when the magnetic pressure $H_1^2/8\pi$ ahead of the simple wave is much less than the hydrostatic pressure p_1 (this is equivalent to satisfying the inequality $U_1 \ll c_1$). In this case

$$r_1 \gg 1, \quad q_{1+} = 1 + U_{1y}^2/c_1^2 + O(U_1^4/c_1^4).$$

Since the quantity q_{1+} diminishes and becomes greater than unity in a self-similar wave, the quantity $\xi \equiv q_+ - 1$ is positive and much smaller than unity. It follows from (3.12) that

$$\Delta_+v_x = -2(c_1-c_2)/(\gamma-1). \tag{4.1}$$

In order to transform (3.13), we rewrite (1.2) as

$$dr\ \ d\xi = \theta(r-1)\xi^{-1} + 2\theta, \qquad \xi \ll 1,$$

hence

$$r = 1 + r_1\xi^{\theta}\xi_1^{-\theta} - \gamma(\gamma-1)^{-1}\xi. \tag{4.2}$$

We denote the smallest value of ξ reached in a complete fast wave by ξ_m. It follows from (4.2) that

$$\xi_m = \xi_1[\xi_1/(\gamma-1)r_1]^{1/(\theta-1)} \ll \xi_1. \tag{4.3}$$

The corresponding value of r is determined from the expression

$$r_m = 1 - \xi_m \approx 1.$$

If $\xi_2 \gg \xi_m$, (3.13) becomes much simpler

$$\Delta_+v_y = \varepsilon c_1\sqrt{\xi_1}\gamma^{-1}r_1^{-1/2\gamma}\,\mathrm{sign}\,(H_xH_{1y}) \times\int\limits_{r_2}^{\infty}r^{-(\gamma+1)/2\gamma}(r-1)^{-(\gamma-1)/\gamma}\,dr. \tag{4.4}$$

The expression (4.4) can be used for any value of ξ_2, since the main contribution to the integral (3.13) is made by values $\xi > \xi_m$. In fact, when $\xi \sim \xi_m$, the value of r is close to unity, and therefore $c_2 \approx c_2r_1^{-(\gamma-1)/2\gamma}$. The contribution to the integral (3.13) from the values $\xi \sim \xi_m$ is given by the formula

$$\Delta'v_y = \frac{c_1}{r_1^{(\gamma-1)/2\gamma}}\int\limits_{\xi_2}^{\xi_c}\frac{[\theta r_1\xi^{\theta-1}\xi_1^{-\theta} - \gamma(\gamma-1)^{-1}]\,d\xi}{[r_1\xi^{\theta-1}\xi_1^{-\theta} - (\gamma-1)^{-1}]^{1/2}},$$

where ξ_c satisfies the inequalities $\xi_m \ll \xi_c \ll \xi_1r_1^{-1/\theta}$. The main contribution to the integral $\Delta'v_y$ is made by the quantities* $\xi \sim \xi_2$. Therefore

$$\Delta'v_y \sim c_1\xi_1^{\gamma/2(\gamma-1)}r_1^{-1/2\gamma(\gamma-1)},$$

which is much less than

$$\Delta_+v_y \sim c_1\xi_1^{1/2}r_1^{-1/2\gamma}.$$

If $r_2 \rightarrow 1$, it follows from (4.4) that

$$\Delta_+v_y = \varepsilon U_{1y}(U_{1x}/c_1)^{1/\gamma}h(\gamma)\,\mathrm{sign}\,(H_xH_{1y}), \tag{4.5}$$

$$h(\gamma) = \frac{\Gamma[(\gamma-1)/2\gamma]\,\Gamma(1/\gamma)}{\gamma\Gamma[(\gamma+1)/2\gamma]}, \qquad h(^5/_3) = 3.52\ldots$$

5. SLOW WAVE WHEN $U_1 \ll c_1$

In front of a slow wave, where $r_1 \gg 1$, the quantity q_{1-} is defined by the expression

$$q_{1-} = r_1^{-1} + O(U_1^4/c_1^4) \ll 1. \tag{5.1}$$

As a result of the decrease in pressure r in a self-similar wave, the inequality $r \gg 1$ is valid only in front of the wave, and relation (5.1) soon ceases to hold. However, since q decreases, the inequality $q \ll 1$ holds over the entire slow wave.

From (1.2) it follows that when $q \ll 1$

$$r = (1+\theta)q_1^{-1} - \theta q^{-1}. \tag{5.2}$$

From (5.2) we find that at the point where cavitation sets in ($r_2 = 0$), the value of q is determined by the relation

$$q_2 = \gamma q_1/2.$$

Using (5.2) we find the discontinuities in the velocity components in a slow wave

$$\Delta_-v_x = -\frac{\varepsilon U_{1x}}{\sqrt{2\gamma}}\int\limits_{\sigma_2}^{1}\frac{\sigma^{-(\gamma+1)/2\gamma}d\sigma}{[1-\sigma(\theta+1)^{-1}]^{1/2}}; \tag{5.3}$$

$$\Delta_-v_y = -\frac{\varepsilon c_1}{\gamma}\,\mathrm{sign}\,(H_xH_{1y})\cdot\int\limits_{\sigma_2}^{1}\sqrt{\frac{1-\sigma(\theta+1)^{-1}}{1-\sigma}}\,\sigma^{-(\gamma+1)/2\gamma}\,d\sigma, \tag{5.4}$$

$\sigma = r/r_1$. If cavitation sets in on a slow wave ($\sigma_2 = 0$), then the expressions for Δ_-v_x and Δ_-v_y are

$$\begin{aligned}\Delta_-v_x &= -\varepsilon U_{1x}f(\gamma),\\ \Delta_-v_y &= -\varepsilon c_1g(\gamma)\,\mathrm{sign}\,(H_xH_{1y}),\end{aligned} \tag{5.5}$$

where

$$f(\gamma) = \frac{1}{\sqrt{2\gamma}}\int\limits_0^1\frac{\sigma^{-(\gamma+1)/2\gamma}}{[1-\sigma(\theta+1)^{-1}]^{1/2}}\,d\sigma, \tag{5.6}$$

$$g(\gamma) = \frac{1}{\gamma}\int\limits_0^1\sqrt{\frac{1-\sigma(\theta+1)^{-1}}{1-\sigma}}\,\sigma^{-(\gamma+1)/2\gamma}\,d\sigma. \tag{5.7}$$

---

*This can be readily verified by comparing the contribution to $\Delta'v_y$ made by the quantities $\xi/\xi_m \rightarrow 1$ and $\xi/\xi_m \gg 1$.

To calculate the quantities $f(\gamma)$ and $g(\gamma)$ it is necessary to expand $[1-\sigma(\theta+1)^{-1}]^{1/2}$ in powers of σ. Integrating term by term, we get

$$f(\gamma)=\sqrt{2\gamma}\left[\frac{1}{\gamma-1}+\frac{1}{2}\frac{1}{\theta+1}\frac{1}{3\gamma-1}\right.$$

$$\left.+\frac{1\cdot 3}{2^3}\frac{1}{(\theta+1)^2}\frac{1}{5\gamma-1}+\ldots\right],$$

$$g(\gamma)=\frac{\sqrt{\pi}\,\Gamma[(\gamma-1)/2\gamma]}{\gamma\Gamma[(2\gamma-1)/2\gamma]}\left[1-\frac{1}{2(\theta+1)}\frac{\gamma-1}{2\gamma-1}\right.$$

$$\left.-\frac{1}{8(\theta+1)^2}\frac{\gamma-1}{2\gamma-1}\frac{3\gamma-1}{4\gamma-1}-\ldots\right].$$

When $\gamma = 5/3$ we have $f(5/3) = 2.78\ldots$ and $g(5/3) = 3.67\ldots$ The transverse magnetic field H_y is given by the relation

$$H_y=\sqrt{8\pi\rho_1(1-\sigma)/\gamma}\,c_1\operatorname{sign}H_y. \tag{5.8}$$

The formulas obtained make it possible to solve the problems of the magnetohydrodynamic piston and of the escape of a magnetohydrodynamic medium into vacuum.

The author expresses his gratitude to A. I. Akhiezer and G. Ya. Lyubarskiĭ for valuable discussions.

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ENERGY AND ANGULAR DISTRIBUTIONS FROM DECAYS OF HYPERNUCLEI

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The energy and angular distribution for the decay $He^4_\Lambda \rightarrow He^3 + p + \pi^-$ is calculated in the impulse approximation. It is shown that by including the interaction in the proton — He^3 system one obtains satisfactory agreement with the experimental data. The energy distribution of π^- mesons from the $He^3_\Lambda \rightarrow d + p + \pi^-$ decay is also discussed briefly.

RECENTLY there has been obtained quite extensive experimental material concerning three-particle π^- meson decays of light hypernuclei.[1,2] It appears that the energy and angular distributions for such decays have a whole variety of interesting properties. Thus, for example, in the overwhelming majority of cases the energy of the π^- mesons lies in a narrow range near the upper limit of the energy spectrum; for He^5_Λ and He^4_Λ one observes a marked asymmetry in the angular distribution, etc. According to the proposal of Dalitz, such a situation is related to the presence of a strong interaction in the system of products of the decay of the hypernucleus.

In the present work, we consider the effect of strong interaction on the energy and angular spectra of the decays

$$He^4_\Lambda \rightarrow He^3 + p + \pi^-, \qquad H^3_\Lambda \rightarrow d + p + \pi^-,$$

and we also make a comparison with similar calculations for the decay of He^5_Λ.[3-6]

The amplitude for the decay of the Λ particle in the hypernucleus has the form

$$M = s + pk_0^{-1}(\boldsymbol{\sigma}\mathbf{k}). \tag{1}$$

Here **k** is the momentum of the π^- meson, k_0 is its value for the decay of a free Λ particle (~101 Mev/c), s and p are the amplitudes for the decay of the Λ particle in the channels with $l = 0$ and $l = 1$ respectively.

We shall neglect the interaction of the π^- mesons with other decay products, since it is small at the energies considered (for more details concerning this, cf. the paper of Byers and Cottingham[6]), and shall take into account only the interaction in the system proton plus residual nucleus. Corrections related to the deformation of the meson cloud of the Λ particle were discussed by Szymanski[5] and were insignificant for light hypernuclei.

Since the binding energy of the Λ particle in the He^4_Λ and H^3_Λ hypernuclei is small, we may assume that the wave function of the hypernucleus is expressed as a product

$$\psi_i = \psi_A \psi_\Lambda(\boldsymbol{\rho}) \chi_i, \tag{2}$$

where ψ_A is the wave function of the nuclear core, $\psi_\Lambda(\rho)$ is a function describing the motion of the Λ-particle with respect to the center of mass of the nuclear core, χ_i is the spin function.

As was shown by Byers and Cottingham,[6] the energy and angular distributions for three-particle decays of hypernuclei are determined by two nontrivial kinematic variables which can be selected to be k, φ or P_Λ, θ (we shall use the notation of reference 6: φ is the angle between the vectors **k** and $\mathbf{k}_f$, where $\mathbf{k}_f$ is the relative momentum of the system proton plus residual nucleus; $\mathbf{P}_\Lambda$ is the vector sum of the momenta of the π^- meson and proton; θ is the angle between the vectors $\mathbf{P}_\Lambda$ and **q**, where **q** is the relative momentum of the π^- meson and proton in their center-of-mass system). In the following we shall use both the variables k, φ, as well as P_Λ, θ.

$He^4_\Lambda \rightarrow He^3 + p + \pi^-$ Decay

Using (1) and (2), we find (after averaging over spin states) for the square of the matrix element of the process, as a function of the spin j of the hypernucleus He^4_Λ, the expression

$$|M_{if}|^2 = s^2 |I^s|^2 + p^2k^2k_0^{-2} |I^t|^2 \quad \text{for } j = 0;$$

$$|M_{if}|^2 = \tfrac{1}{3} p^2k^2k_0^{-2} |I^s|^2 + (s^2 + \tfrac{2}{3} p^2k^2k_0^{-2}) |I^t|^2 \quad \text{for } j = 1. \tag{3}$$

Here

$$I^{t,s} = \int \psi^{t,s*}(\boldsymbol{\rho}) e^{-i3\mathbf{k}\boldsymbol{\rho}/4} \psi_\Lambda(\boldsymbol{\rho})\, d\boldsymbol{\rho},$$

$$\psi^{t,s}(\boldsymbol{\rho}) = \frac{1}{2} \sum_{l=0}^{\infty} i^l (2l+1) [h_l^{(1)}(k_f\rho) + \exp(-2i\delta_l^{t,s}) h_l^{(2)}(k_f\rho)] P_l(\cos\gamma), \tag{4}$$

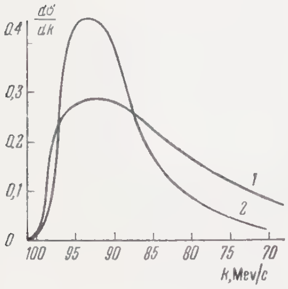

FIG. 1. Energy distribution of π^- mesons in the decay of He^4_Λ: 1 – without interaction in the final state; 2 – including interaction.

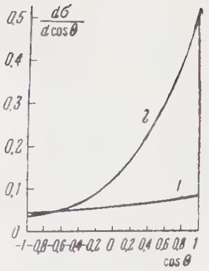

FIG. 2. Angular distribution of π^- mesons from the decay of He^4_Λ. The notation is the same as for Fig. 1.

$h_l^{(1)}$ and $h_l^{(2)}$ are spherical Hankel functions of the first and second kind.

The problem of finding the phases $\delta_l^{t,s}$ for scattering of protons on He^3 nuclei was considered by Bransden and Robertson,[7] who started from a potential for the interaction between the two nucleons without including spin-orbit or tensor forces. As the comparison of the results of their calculations with experimental data[8] showed, in the case of Serber forces one obtains satisfactory agreement with experiment for proton energies up to 10 Mev. We shall therefore use the phase shifts calculated by Bransden and Robertson for Serber-type interaction, and shall include only states with $l = 0$ and $l = 1$, since phase shifts with $l \geq 2$ are small at the energies considered here.

For the function $\psi_\Lambda(\rho)$ we shall choose the expression

$$\psi_\Lambda(\rho) \sim \frac{1}{\rho}\left(e^{-\alpha\rho} - e^{-\beta\rho}\right).$$

By means of a variational procedure similar to that used by Derrick,[9] we find that $\beta = 3.8\alpha$. The Coulomb interaction is included in the same way as in the work of Tang.[3] In accordance with the result of Dalitz,[10] the spin j = 0 was chosen for the hypernucleus He^4_Λ.

In Fig. 1 we show the energy spectrum of the π^- mesons (in arbitrary units) which has the form

$$\frac{d\sigma}{dk} \sim \int_{-1}^{+1} |M_{if}(k, \cos\varphi)|^2 k^2 k_f \, d\cos\varphi. \qquad (5)$$

Curve 1 was calculated without interaction in the proton – He^3 system, and curve 2 including such interaction. Both curves were calculated on the assumption that | p/s | = 1.[10]

As the experimental data[1,2] show, in almost all the decays the momenta of the π^--mesons lie in a narrow range of momenta from 80–100 Mev/c (there are no more than 10% of all the decays outside this interval). The energy spectrum calculated without including final state interaction (curve 1) shows that in this case, for approximately 35% of all the decays, the momenta of the π^- mesons will lie below 80 Mev/c, in contradiction with the experimental data. The spectrum, calculated including such an interaction, shows that in approximately 90% of the cases the momenta of the π^- mesons should lie in this interval, i.e., agreement with the experimental data is satisfactory.

In Fig. 2 we show the angular distribution

$$\frac{d\sigma}{d\cos\theta} \sim \int_0^{P_{\Lambda max}} |M_{if}(P_\Lambda, \cos\theta)|^2 P_\Lambda^2 q \, dP_\Lambda. \qquad (6)$$

The comparison of the calculated angular distributions with the experimental data[1,2] clearly shows the presence of strong interaction in the system of decay products. Curve 1 (no interaction) increases very slowly with increasing cos θ. The number of decays for which $\cos\theta < 0$ is ~45% which sharply contradicts the forward asymmetry observed experimentally in the decay of He^4_Λ. On the other hand, when we include interaction in the proton – He^3 system (curve 2), the number of decays for which $\cos\theta < 0$ is 15%, which is in satisfactory agreement with the experimental data.

We note that the energy and angular distributions presented in Figs. 1 and 2 depends very slightly on the value of the variational parameter β in the wave function $\psi_\Lambda(\rho)$ (cf. the paper of Picasso and Rosati[12]). Including the interaction of the π^- mesons with other decay products also does not essentially change our results.

Several authors,[3,5] especially and in detail Byers and Cottingham,[6] have discussed the energy and angular distribution from the $He^5_\Lambda \rightarrow He^4 + p + \pi^-$ decay. Byers and Cottingham[6] came to the conclusion that these distributions (and especially the angular distribution) are essentially determined by the resonance character of the phase shifts for scattering of protons by He^4_Λ. However, our calculations do not confirm the conclusion that the presence of resonances plays such an important role (cf. also reference 5), at least for the decays of light hypernuclei. As we know, the phase shifts for scattering of proton by He^3 do not have resonance character,[7,8] but nevertheless the angular distribution calculated using them for the $He^4_\Lambda \rightarrow He^3 + p + \pi^-$ decay has a marked asymmetry, in agreement with experiment.

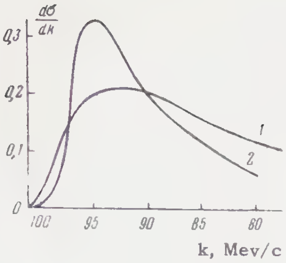

FIG. 3. Energy distribution of π^- mesons from the decay of He^3_Λ. The notation is the same as for Fig. 1.

The $H^3_\Lambda \rightarrow d + p + \pi^-$ Decay

The square of the matrix element of the decay for a spin j = ½ of the H^3_Λ hypernucleus has the form

$$|M_{if}|^2 = (s^2 + \tfrac{1}{9} p^2 k^2 k_0^{-2}) |I^d|^2 + \tfrac{8}{9} p^2 k^2 k_0^{-2} |I^q|^2. \quad (7)$$

Here

$$I^{d,q} = \int \psi^{d,q*}(\boldsymbol{\rho}) e^{-2ik\rho/3} \psi_\Lambda(\boldsymbol{\rho}) d\boldsymbol{\rho}. \quad (8)$$

The function $\psi^{d,q}(\rho)$ is obtained from (4) by replacing the phase shifts $\delta_l^{t,s}$ by the phases $\delta_l^{d,q}$, which were calculated by Massey.[11] For the function $\psi_\Lambda(\rho)$ we chose the expression[12]

$$\psi_\Lambda(\rho) \sim e^{-\gamma\rho}, \qquad \gamma = 0.75. \quad (9)$$

Using (7) — (9) we computed the energy spectrum of the π^- mesons, which is shown in Fig. 3. Curve 2, computed including interaction in the proton-deutron system, shows that ~80% of all the decays are contained in the interval 85 — 100 Mev/c, which is in satisfactory agreement with experimental data.[13]

Thus the results of our computations show that the effect of final-state interaction essentially determines the shape of the energy spectrum and angular distribution for decays of hypernuclei, even when this interaction is described by non-resonant phase shifts.

In conclusion, I express my gratitude to Prof. D. D. Ivanenko for support in carrying out the present work, and also to N. S. Il'ina for carrying out the numerical computations.

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PION-PION INTERACTION AND THE ELECTROMAGNETIC STRUCTURE OF THE NUCLEON IN THE STATIC THEORY

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The mean square radius of the pion charge distribution is estimated on the basis of the relationship between this radius and the pion-pion scattering resonance energy derived from the dispersion relations. The mean square radius of nucleon charge is calculated within the framework of the Chew-Low-Wick static theory by taking the electromagnetic structure of the meson into account, and good agreement with experimental data is obtained.

1. INTRODUCTION

THE investigation of the pion-pion interaction aroused great interest in recent years after it became evident that knowledge concerning this interaction is required for further progress in the study of strong interactions.[1] Chew and Mandelstam[2] have studied the pion-pion interaction using a double dispersion representation, and have concluded that the amplitude of pion-pion scattering must exhibit resonance in the state with total angular momentum and isotopic spin both equal to unity. Since the amplitude of pion-pion scattering is still unknown because it is impossible to perform a direct scattering experiment on a system of unstable particles, it is interesting to consider every possible procedure for attempting to determine the basic parameters of this amplitude by analyzing the contributions of the pion-pion interaction to different experimentally observable effects.

Frazer and Fulco[3] have shown how to take into account the contribution of the pion-pion interaction to the isotopic-vector form factor of a nucleon. However, their work is not entirely correct. For example, their solution for the annihilation amplitude of a nucleon-antinucleon pair [Eq. (4)] is wrong, as can be seen by comparing the imaginary part of the solution in Eq. (4) for $s > 4\mu^2c$ with the corresponding imaginary part of the exact solution.[4] Another shortcoming lies in the fact that the expression for the spectral density neglects corrections for secondary pion-nucleon scattering, which plays an essential part.[5]

2. MEAN SQUARE PION CHARGE RADIUS

We shall point out a simple possibility of relating the resonance energy in the pion-pion scattering amplitude to the mean square pion charge radius. We know that the dispersion equation (in subtracted form) for the meson form factor G, after making the reasonable assumption that the largest contribution comes from two-meson intermediate states, has the solution[6]

$$G(s) = \exp\left\{\frac{s}{\pi}\int_{4\mu^2}^{\infty}\frac{\delta(s')\,ds'}{s'(s'-s-i\varepsilon)}\right\}, \tag{1}$$

where $\delta(s)$ is the pion-pion scattering phase in the state with $J = I = 1$ when the total energy in the center-of-mass system is $\sqrt{s}$. (1) leads to a mean square charge radius represented by

$$\langle r^2\rangle_\pi = 6\left.\frac{\partial G(s)}{\partial s}\right|_{s=0} = \frac{6}{\pi}\int_{4\mu^2}^{\infty}\frac{\delta(s')\,ds'}{s'^2}. \tag{2}$$

We know nothing concerning the energy dependence of the phase. Assuming that resonance exists in the state of present interest and that this resonance is quite narrow (i.e., that the scattering amplitude is large only in the resonance region), the behavior of the phase can be approximated by the function

$$\delta(s) = \pi\theta(s - s_0), \tag{3}$$

where s_0 is the square of the resonance energy and $\theta(s)$ is a step function that vanishes for negative values of its argument. We may expect that the total error associated with the approximation (3) will be small even when the resonance is not very narrow. Integration leads to the following relationship between the mean square pion charge radius and the square of the resonance energy:

$$\langle r^2\rangle_\pi s_0 = 6. \tag{4}$$

Using the tentative value $s_0 = 22\mu^2$ for the resonance energy,[7] we obtain

$$(\langle r^2\rangle_\pi)^{1/2} = 0.52\,\mu^{-1} = 0.73 \text{ f}.$$

It is necessary to verify the extent to which this unexpectedly large value of the pion radius is consistent with the experimentally observed nucleon charge distribution.

Riazuddin[8] has calculated the mean square pion charge radius by assuming equality of the experimental and the theoretical mass difference between neutral and charged pions. This work was limited to only a one-meson intermediate state in the expression for the amplitude of a virtual Compton effect for the pion. However, in view of the resonance character of the pion-pion interaction the virtual photoproduction amplitude is not small, so that two-meson intermediate states cannot be neglected. Therefore the result $(\langle r^2\rangle_\pi)^{1/2} = 0.5$ f must be regarded as the lower limit of the pion radius.

3. ELECTROMAGNETIC STRUCTURE OF THE NUCLEON

For the purpose of estimating the role of pion-pion interaction in the electric charge structure of nucleons, the vector part of the mean square nucleon charge radius was calculated in the static theory of Chew, Low, and Wick.[9] We know that with a suitable cutoff constant this theory yields the same basic results as local-field theory with dispersion relations.[5,10]

Although the dispersion-relation method provides a more consistent approximation, and takes, for example, relativistic corrections into account, it requires that we know the pion-nucleon scattering amplitude for non-physical momentum transfers $s_3 > 4\mu^2$. Because of the strong pion-pion interaction, this amplitude is not analytic with respect to s_3 for $s_3 > 4\mu^2$, and cannot be continued outside of the physical region by expansion in Legendre polynomials. The problem of analytic continuation does not arise in the static theory, which from this point of view yields more reliable results than the dispersion relations.

Earlier calculations using both the static and the dispersion theory yield comparatively acceptable results in the approximation where the pion-nucleon scattering amplitude is represented by a one-nucleon term.[10-12] However, this agreement with experiment must be regarded as accidental; it was shown in reference 5 that consideration of secondary pion-nucleon scattering seriously changes the results and leads to a very small proton charge radius that conflicts with the experimental results obtained by Hofstadter's group. The same authors pointed out the necessity of taking into account the pion electromagnetic structure, which they estimated by means of perturbation theory.

In our case the pion-pion resonance interaction is partially accounted for phenomenologically through the introduction of an electromagnetic form factor for the pion. The pion-nucleon scattering amplitude is effectively corrected by means of a suitable cutoff parameter. $v(k) = L^2/(L^2 + k^2)$ was used as the cutoff function; the cutoff parameter $L = 6.5\mu$ was chosen to give the correct value for the vector part of the anomalous nucleon magnetic moment. Secondary scattering terms were subject to the same assumptions as in reference 13, where small phase shifts of the pion-nucleon scattering amplitude were calculated.

A similar calculation was performed for the mean charge of the meson cloud, which determines the degree Q of virtual dissociation of the nucleon into a nucleon and charged pion. We obtained $Q = 23\%$, which results from a semi-phenomenological analysis of experimental data on nucleon structure when a physical nucleon is regarded as a system consisting of a nucleonic core and a virtual pion cloud.

For the vector part of the mean square of the nucleon charge radius we obtained

$$\langle r^2\rangle_V = 0.047\ \mu^{-2} + 0.063\ \mu^{-2} + 0.039\ \mu^{-2} = 0.149\ \mu^{-2};$$

the first term accounts for the charge distribution of the cloud of virtual point pions; the second term, which is $Q\langle r^2\rangle_\pi$, is the contribution of the meson electromagnetic structure. The last term, which is $(3 - 6Q)[\partial^2 v(k)/\partial k^2]_{k=0}$, takes into account the charge distribution of the meson core as well as the contributions of virtual high-energy pions and the possible contributions of other heavier particles.

It is impossible at present to calculate the scalar part of the nucleon form factor, since this part results from a photon-nucleon interaction through a three-meson intermediate state. Amplitudes corresponding to diagrams with three external meson lines, knowledge of which is required for such calculations, have thus far not been investigated. We may assume, however, that the vector and scalar parts of the mean square nucleon charge radius are equal, as is indicated by the experimentally observed vanishing of the neutron electric charge. The rms radius of the proton charge then becomes

$$(\langle r^2\rangle_p)^{1/2} = 0.77 \text{ f}$$

in good agreement with experimental data.

4. CONCLUSION

We have thus shown that the hypothesis of a pion-pion resonance interaction makes it possible for the static theory to describe experimental data regarding the mean square nucleon charge radius if resonance occurs for $s_0 = 22\mu^2$.[7] Our results indicate, furthermore, that corrections for secondary pion-nucleon scattering, which had previously been considered small,[10-12,14] actually played an important part, as has been confirmed in reference 5. It should be noted that the point pion cloud, which has thus far been regarded as most important, actually plays only a small part. On the other hand, the electromagnetic structure of the pion is very important. The static theory may be corrected through the use of dispersion relations, but this evidently requires a previous solution of Mandelstam's equations.

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ON THE PROBLEM OF THE KNIGHT SHIFT IN SUPERCONDUCTORS

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The problem of the Knight shift in superconductors of small size is discussed. It is shown that the derivation of a finite value of the shift at absolute zero obtained in the papers of Schrieffer[3] and of Anderson[4] is incorrect.

RECENTLY a number of papers[1-4] has been devoted to the theory of the so-called Knight shift in superconductors, i.e., the change in the nuclear resonance frequency as compared to dielectrics. The aim of these papers is to explain the experimental data obtained by Reif for superconducting emulsions.[5]

As is well known, the cause of the Knight shift in metals is the paramagnetism of the conduction electrons. Because the electron wave function is anomalously large in the neighborhood of the nucleus, the magnetization of the electrons changes the magnetic field acting on the nucleus. The difference between the effective and the external field has the following form

$$\Delta H = (8\pi/3N_{\mathrm{at}})\,|\psi(0)|^2\,\chi H, \qquad (1)$$

where $|\psi(0)|^2$ is the probability density of finding the electron at the position of the nucleus, N_{at} is the number of atoms per unit volume, χ is the paramagnetic susceptibility of the electrons, and H is the external field.

Since the conduction electron states are altered in superconductors, one should expect a change in the magnitude of the Knight shift. Formula (1) is still valid in this case, with the susceptibility evidently undergoing the principal change in comparison with the normal metal. The observation of this effect is made difficult by the fact that the field does not penetrate deeply into a bulky superconductor. A homogeneous field can be produced only in superconductors of dimensions much smaller than δ, the penetration depth for a static field. It is just for this reason that an emulsion is utilized in Reif's experiments.

In Yosida's paper[1] a bulky superconductor in a homogeneous field was considered. Such a formulation of the problem does not correspond to actual experimental conditions. In order to be able to completely leave out of account the effect of the boundary conditions and of the inhomogeneity of the field it is necessary that the depth of penetration of the field and the dimensions of the sample should be large compared to the principal parameter of the theory $\xi_0 \sim \hbar v/kT_c$, the characteristic correlation radius. Only in such a case could Yoshida's results, obtained for an infinite superconductor, be used for the interpretation of Reif's experiments. However, it is well known that for the majority of superconductors δ is considerably smaller than ξ_0. From the formal point of view the calculations of the susceptibility in reference 1 for a bulk sample are of course correct, since in order to calculate the coefficient of proportionality between the magnetic moment and the field (assuming the latter to be homogeneous) it is not necessary to solve the problem of the penetration of the field into the sample. We are here simply discussing the fact that actually the field remains homogeneous at distances smaller than $\delta \ll \xi_0$. The attempt by Martin and Kadanoff[2] to take into account the inhomogeneity of the field in the sample cannot lead to any improvement, since such an inhomogeneity increases first of all the line width.

The problem can be formulated quite correctly for superconducting alloys. If the concentration of the impurities is sufficiently great then the role of the correlation parameter is played by the mean free path l. Under these conditions the situation described by London ($\delta \gg l$) arises in the superconductor, and it becomes possible to regard the field in the sample as homogeneous.

As has been pointed out earlier,[7,8] actual samples of small dimensions are conglomerates of crystallites with dimensions of the order of or smaller than the sample dimensions. Because of this, such samples have properties which are rather close to superconducting alloys with an effective mean free path of the same order of magnitude. The paramagnetic susceptibility of superconducting alloys has been recently investigated by Schrieffer[3] and by Anderson.[4] In particular, in these investigations the conclusion was reached that the presence of impurities leads to a finite Knight shift at $T = 0$ which is in agreement with the extrapolation of Reif's results to $T = 0$. In this note it will be shown that such a conclusion is erroneous.

The method for the study of superconductors containing impurities has been developed by us previously.[7,8] In the present case it is necessary to evaluate the spin magnetic moment of the system of electrons in a homogeneous magnetic field. It is given by

$$\mathbf{M} = \mu_0 \operatorname{Sp}\left[\exp\left(\frac{\Omega + \hat{N}\mu - \hat{\mathscr{H}}}{T}\right)\psi_\alpha^+(x)\,\sigma_{\alpha\beta}\,\psi_\beta(x)\right], \tag{2}$$

where μ_0 is the Bohr magneton and σ are the Pauli matrices. The Hamiltonian for the interaction between the electron spins and the magnetic field has the form

$$\hat{\mathscr{H}}_1 = -\mu_0 \int \psi_\alpha^+(\mathbf{x})\,(\sigma_{\alpha\beta}\mathbf{H})\,\psi_\beta(\mathbf{x})\,d^3\mathbf{x}. \tag{3}$$

By utilizing the techniques used in quantum field theory at finite temperatures[8,9] and by restricting ourselves to terms of the first order in H we obtain

$$\mathbf{M} = -\mu_0^2 \lim_{\substack{\tau' \to \tau + 0 \\ x' \to x}} \int_0^{1/T} d\tau_y \int d^3\mathbf{y}\, \operatorname{Sp}\left[\exp\left(\frac{\Omega + \hat{N}\mu - \mathscr{H}_0}{T}\right)\right.$$
$$\left.\times T_\tau\left(\psi_\beta(x)\,\psi_\gamma^+(y)\,\psi_\delta(y)\,\psi_\alpha^+(x')\right)\sigma_{\alpha\beta}\,(\sigma_{\gamma\delta}\mathbf{H})\right] \tag{4}$$

In this formula the operators ψ are not free, but include the interaction with the impurities:

$$\hat{\mathscr{H}}_2 = \int \psi_\alpha^+(\mathbf{x})\,\psi_\alpha(\mathbf{x}) \sum_a V(\mathbf{x} - \mathbf{x}_a)\,d^3\mathbf{x}, \tag{5}$$

where $\mathbf{x}_a$ is the position vector of the impurity atom.

Further calculations have been carried out in complete analogy with the way this was done in references 7 and 8. It was shown there that averaging over the coordinates of the impurity atoms leads to the introduction of a special diagram technique. Each impurity line in a diagram arises from averaging two factors corresponding to scattering by one of the impurity atoms. It is sufficient to retain only the non-intersecting lines in order to obtain a result which differs from the correct one by small terms only. If we apply this procedure to formula (4) then the calculation of the magnetic moment obviously reduces to the evaluation of the sum of loops with two spinor vertices $\sigma_{\alpha\beta}$ and with all the nonintersecting impurity lines.

In averaging over the impurity coordinates we must take into account the fact that, for example, $\overline{G(x, y)\,G(y, x)}$ is not equal to $\overline{G(x, y)} \cdot \overline{G(y, x)}$. In the opposite case this expression would simply be given by

$$G_0(x - y)\,G_0(y - x)\exp(-|\mathbf{x} - \mathbf{y}|/l),$$

where l is the mean free path, while $G_0(x - y)$ is the Green's function for the pure superconductor. This incorrect assumption is the one made in Schrieffer's note[3] and leads to an erroneous result.

If we denote by $\Pi^1_{\beta\alpha}(\mathbf{p}, \omega_n)$ the Fourier component of

$$\int_0^{1/T} d\tau_y \int d^3\mathbf{y}\, \operatorname{Sp}\left[\exp\left(\frac{\Omega + \hat{N}\mu - \hat{\mathscr{H}}_0}{T}\right)\right.$$
$$\left.\times T_\tau\left(\psi_\beta(x)\,\psi_\gamma^+(y)\,\psi_\delta(y)\,\psi_\alpha^+(x')\right)\sigma_{\gamma\delta}\right],$$

we can easily relate this quantity by means of appropriate equations to Fourier components of three other similar quantities:

$$\int d^4y\,\overline{[\psi_\beta(x)\,\psi_\gamma^+(y)\,\psi_\delta(y)\,(\psi(x')\,\hat{g})_\alpha]}\,\sigma_{\gamma\delta},$$

$$\int d^4y\,\overline{[(\hat{g}\psi^+(x))_\beta\,\psi_\gamma^+(y)\,\psi_\delta(y)\,\psi_\alpha(x')]}\,\sigma_{\gamma\delta},$$

$$\int d^4y\,\overline{[(\hat{g}\psi^+(x))_\beta\,\psi_\gamma^+(y)\,\psi_\delta^+(y)\,(\psi(x')\,\hat{g})_\alpha]}\,\sigma_{\gamma\delta},$$

where

$$\hat{g} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The magnetic moment is expressed in terms of $\Pi^1_{\beta\alpha}$ by means of the relation

$$\mathbf{M} = -\mu_0^2\,\sigma_{\alpha\beta}\,T \sum_n (2\pi)^{-3} \int \mathbf{H}\Pi^1_{\beta\alpha}(\omega_n, \mathbf{p})\,d^3\mathbf{p}, \tag{6}$$

where $\omega_n = \pi T(2n + 1)$.

Since all these calculations corespond completely to those carried out in reference 7, we shall not dwell on the solution of these equations, and shall simply state the answer:

$$\chi = 2\mu_0^2\,(mp_0/2\pi^2)\,(N_n/N) = \chi_n N_n/N, \tag{7}$$

where χ_n is the susceptibility of the normal metal, and N_n/N is the ratio of the number of "normal" electrons to the total number of electrons given by

$$\frac{N_n}{N_0} = \frac{1}{2T} \int_{\Delta(T)}^{\infty} \frac{\varepsilon\,d\varepsilon}{\sqrt{\varepsilon^2 - \Delta^2(T)}\cosh^2(\varepsilon/2T)}. \tag{8}$$

The function $N_s = N - N_n$ determines the penetration depth for a pure superconductor of the London type.

Formulas (7) and (8) agree with Yosida's result[1] obtained formally for the case of a bulk of pure superconductor in a homogeneous field. This is explained by the fact that these formulas do not contain the mean free path. We also note that in the case considered here $l \gg 1/p_0$ the size of the gap $\Delta(T)$ agrees with the same quantity for the case of a pure superconductor.

From our results it follows, in particular, that for $T = 0$ the susceptibility χ vanishes, i.e., that no Knight shift should be observed. This conclusion is perfectly natural. In its ground state the superconductor has no magnetic moment, and the excited state is separated from the ground state by an energy gap. This applies not only to pure superconductors, but also to alloys with nonmagnetic impurities. Nevertheless, this result contradicts Reif's experiments,[5] which, as has been noted by Yosida, give values of χ larger than those given by formula (7). The quantity χ does not vanish when we extrapolate to $T = 0$.

In this connection it is necessary to subject to some criticism the interpretation of Reif's experiment. As we have already mentioned, in these experiments emulsions were used, i.e., an aggregate of small spherical samples of different sizes. Apparently at the present time this is the only method for investigating nuclear resonance in superconductors, and it can give rise to no objections if the particles are sufficiently small. In this case they had a diameter d on the average equal to one third of the penetration depth. However, the distribution of particles with respect to size is fairly broad, and since the effectiveness of individual particles is proportional to their volume the most important role is played by the large particles, even though they are less numerous. In such particles the field is inhomogeneous, which is in complete agreement with the large width of the resonance line obtained by Reif at low temperatures.

One other essential circumstance should also be noted. As is well known, the transition from the superconducting to the normal state in particles of dimensions smaller than the penetration depth is a phase transition of the second kind for which the superconducting "gap" in the spectrum vanishes in a continuous manner. This means that particles for which the external magnetic field is only slightly lower than the critical value (we recall that $H_c \sim 1/d$) will be very close in their properties to the normal metal right down to very low temperatures. From this it follows that in the given external field (which, incidentally, was approximately equal to 1000 gauss) the particles cannot be separated into completely normal ones and totally superconducting ones with corresponding resonance frequencies, but give rise to various intermediate values of the resonance frequency.

In view of the foregoing, it appears to us that at the present time it is premature to speak of the existence of a contradiction between theory and experiment.

In conclusion we express our gratitude to Academician L. D. Landau for discussion of this work.

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ON THE INFRARED CATASTROPHE IN SCALAR QUANTUM ELECTRODYNAMICS

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The asymptotic behavior of the four-vertex function in the infrared region is studied in the quantum electrodynamics of particles with zero spin. A procedure for removing the infrared divergences and summing the probabilities is discussed for the process of scattering of charged mesons by charged mesons with the emission of an arbitrary number of soft quanta.

1. In spinor electrodynamics the methods for finding the infrared asymptotic behavior have been studied thoroughly.[1-4] In the electrodynamics of spinless particles there is an additional complication owing to the necessity of including the four-boson interaction.

As has been pointed out previously,[5,6] the group of multiplicative renormalizations in scalar electrodynamics has two invariant charges: e^2d and $hd_M^2\square_1$, where d, d_M, and $\square_1$ are the scalar functions that appear respectively in the transverse Green's function of the photon, in the Green's function of the meson, and in the part of the retarded interaction of mesons with the matrix structure $X_{\alpha\beta}X_{\gamma\delta}$ [X is the Kemmer matrix (cf. reference 5)]. In the infrared region the photon propagation function is regular, and therefore the invariant charge that describes the electromagnetic interaction is a constant.[2]

The situation for the four-boson interaction is more complicated. As has been shown by Gor'kov,[7] the Green's function of the scalar meson has an infrared singularity. If it turns out that this is not compensated by the singularities of the function $\square_1$ in the expression for the second invariant charge, then to find the correct asymptotic behavior of any function that depends on h one will have to take into account the behavior of the quantity $hd_M^2\square_1$ in this region. As for the meson Green's function and the vertex part, their asymptotic expressions do not depend on the second charge h.[5,6]

The situation is different in the treatment of diagrams with four external meson lines. Let us first find their behavior in the infrared region in the low orders of perturbation theory. We confine ourselves to those parts of the diagrams of the retarded interaction of mesons for which the matrix structure is determined by a direct product of X matrices. For such a structure infrared singularities can be given only by the diagrams shown in Fig. 1. The required expressions are obtained by the calculation of the corresponding Feynman integrals having singularities at $k \sim 0$ (k is the variable of integration) when the squares of the external four-momenta simultaneously approach m^2.

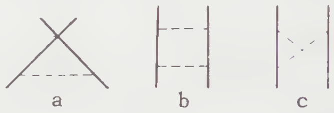

FIG. 1

The calculation is made with the photon Green's function adjusted to $d_l^0 = 1$. One then gets different expressions for the index of the infrared singularity, depending on whether the particles involved in the scattering process have like or unlike charges.

We have

$$\square_1^{(2)}\left(\frac{p^2-m^2}{m^2},\ s_1, s_2, s_3, e^2, h\right) = 1 + \left[e^2 f_1(s_1, s_2, s_3) + \frac{e^4}{h} f_2(s_1, s_2, s_3)\right] \ln\frac{p^2-m^2}{m^2} + C_M, \tag{1}$$

where C_M is a finite arbitrary constant, the uncertainty to within which $\square_1$ is calculated. For the scattering of particles with like charges

$$f_1(s_1, s_2, s_3) = \frac{1}{4\pi^2}\left\{-\frac{s_1 - {}^1/_2}{\sqrt{s_1(s_1-1)}}\ln\frac{1+\sqrt{(s_1-1)/s_1}}{1-\sqrt{(s_1-1)/s_1}} + \frac{|s_2| + {}^1/_2}{\sqrt{|s_2|(1+|s_2|)}}\ln\frac{1+\sqrt{|s_2|/(1+|s_2|)}}{1-\sqrt{|s_2|/(1+|s_2|)}} + \frac{|s_3| + {}^1/_2}{\sqrt{|s_3|(1+|s_3|)}}\ln\frac{1+\sqrt{|s_3|/(1+|s_3|)}}{1-\sqrt{|s_3|/(1+|s_3|)}}\right\},$$

$$f_2(s_1, s_2, s_3) = \frac{1}{16\pi^2}\left\{-\frac{(s_1 - {}^1/_2)^2}{\sqrt{s_1(s_1-1)}}\frac{s_2+s_3}{4s_2s_3}\ln\frac{1+\sqrt{(s_1-1)/s_1}}{1-\sqrt{(s_1-1)/s_1}} + \frac{(|s_2| + {}^1/_2)^2}{4s_3\sqrt{|s_2|(1+|s_2|)}}\ln\frac{1+\sqrt{|s_2|/(1+|s_2|)}}{1-\sqrt{|s_2|/(1+|s_2|)}} + \frac{(|s_3| + {}^1/_2)^2}{4s_2\sqrt{|s_3|(1+|s_3|)}}\ln\frac{1+\sqrt{|s_3|/(1+|s_3|)}}{1-\sqrt{|s_3|/(1+|s_3|)}}\right\};$$

for the scattering of particles with opposite charges the expressions for f_1 and f_2 are obtained from Eqs. (2a) and (2b) by the interchanges $s_1 \rightleftarrows s_3$, $s_2 \rightarrow s_2$. Here

$$s_1 + s_2 + s_3 = 1, \qquad s_1 > 1, \qquad s_2 < 0, \qquad s_3 < 0; \tag{3}$$

$$s_1 = (p+q)^2/4m^2, \qquad s_2 = (p'-p)^2/4m^2, \qquad s_3 = (p'-q)^2/4m^2; \tag{4}$$

p and q are the initial four-momenta of the particles, and p′ and q′ are the final four-momenta.

Thus it turns out that the four-meson function $\square_1$ has a logarithmic singularity when the squares of the external momenta approach m^2. Furthermore, the coefficient of this singularity depends on the two independent momenta. Finally, an essential fact is that $\square_1$ depends on h, so that in the summation of the maximum powers of the divergent logarithms it is important to take account of the behavior of the invariant $hd_M^2 \square_1$ in the infrared region.

2. Let us derive the functional and differential equations for finding the infrared asymptotic behavior of the function $\square_1$. Because of the existence of the condition (3) and the dependence of the coefficient of the logarithm on the momenta s_i it is impossible to normalize $\square_1$ to the same normalization momentum for all three independent squares of momenta. Therefore the normalization condition for $\square_1$ must be chosen in the form

$$\square_1\left(\frac{p^2-m^2}{m^2},\ s_1,\ s_2,\ s_3,\ e^2,\ h\right)\Bigg|_{\substack{p^2=\lambda^2\\ s_i=s_i^{(0)}}} = 1. \tag{5}$$

Using the given condition to fix the arbitrariness contained in $\square_1$, we write the function normalized to unity in the form

$$\square_1 = \square_1\left(\frac{p^2-m^2}{m^2},\ \frac{\lambda^2-m^2}{m^2},\ s_1,\ s_2,\ s_3,\ s_1^{(0)},\ s_2^{(0)},\ s_3^{(0)},\ e^2,\ h\right), \tag{6}$$

so that the condition (5) takes the form

$$\square_1\left(\frac{\lambda^2-m^2}{m^2},\ \frac{\lambda^2-m^2}{m^2},\ s_i^{(0)},\ s_i^{(0)},\ e^2,\ h\right) = 1. \tag{5a}$$

Using the fact that in the infrared region d = 1, we get a functional equation for the normalized $\square_1$:

$$\begin{aligned}\square_1&\left(\frac{p^2-m^2}{m^2},\frac{\lambda^2-m^2}{m^2},s_i,s_i^{(0)},e^2,h\right)\\ &=\square_1\left(\frac{\lambda'^2-m^2}{m^2},\frac{\lambda^2-m^2}{m^2},s_i^{(1)},s_i^{(0)},e^2,h\right)\\ &\times\square_1\left(\frac{p^2-m^2}{m^2},\frac{\lambda'^2-m^2}{m^2},s_i,s_i^{(1)},e^2,h'\right),\end{aligned} \tag{7}$$

where

$$\begin{aligned}h' = hd_M^2&\left(\frac{\lambda'^2-m^2}{\lambda^2-m^2},e^2,h\right)\\ &\times\square_1\left(\frac{\lambda'^2-m^2}{m^2},\frac{\lambda^2-m^2}{m^2},s_i^{(1)},s_i^{(0)},e^2,h\right),\end{aligned} \tag{8}$$

λ'^2, $s_i^{(1)}$ is the normalization point for the second function $\square_1$ in the right member of Eq. (7). The relations (7) and (8) are obtained by substitution of Eq. (6) into the system of equations of the renormalization group, Eq. (2) of reference 5.

The functional equation (7) is very cumbersome. To simplify it, we represent $\square_1$ in the following way:

$$\begin{aligned}\square_1&\left(\frac{p^2-m^2}{m^2},\frac{\lambda^2-m^2}{m^2},s_i,s_i^{(0)},e^2,h\right)\\ &=\square_1\left(\frac{\lambda^2-m^2}{m^2},\frac{\lambda^2-m^2}{m^2}s_i,s_i^{(0)},e^2,h\right)\\ &\times\square_1\left(\frac{p^2-m^2}{m^2},\frac{\lambda^2-m^2}{m^2},s_i,s_i,e^2,h_\alpha\right),\end{aligned} \tag{9}$$

where

$$h_\alpha = h\square_1\left(\frac{\lambda^2-m^2}{m^2},\frac{\lambda^2-m^2}{m^2},s_i,s_i^{(0)},e^2,h\right),$$

$$d_M(1,e^2,h) = 1.$$

The equation (9) can be obtained by setting $\lambda'^2 = \lambda^2$, $s_i^{(1)} = s_i$ in the right member of Eq. (7). The convenience of this representation is that the factors in the right member of Eq. (9) are separately normalized to unity, the first for $s_i = s_i^{(0)}$ and the second for $p^2 = \lambda^2$. Therefore in virtue of the fact that $\square_1$ is a homogeneous function of the momentum arguments we can always set

$$\square_1\left(\frac{p^2-m^2}{m^2},\frac{\lambda^2-m^2}{m^2},s_i,s_i,e^2,h_\alpha\right)\equiv\widetilde{\square}_1\left(\frac{p^2-m^2}{\lambda^2-m^2},s_i,e^2,h_\alpha\right),$$

so that Eq. (9) takes the form

$$\begin{aligned}\square_1&\left(\frac{p^2-m^2}{m^2},\frac{\lambda^2-m^2}{m^2},s_i,s_i^{(0)},e^2,h\right)\\ &=\square_1\left(\frac{\lambda^2-m^2}{m^2},\frac{\lambda^2-m^2}{m^2},s_i,s_i^{(0)},e^2,h\right)\\ &\times\widetilde{\square}_1\left(\frac{p^2-m^2}{\lambda^2-m^2},s_i,e^2,h_\alpha\right).\end{aligned} \tag{10}$$

We emphasize that the infrared singularity is entirely contained in $\widetilde{\square}_1$.

Representations analogous to Eq. (10) can also be obtained for the other two functions $\square_1$ involved in Eq. (7). Making the respective replacements $p^2 \rightarrow \lambda'^2$, $s_i \rightarrow s_i^{(1)}$ and $\lambda^2 \rightarrow \lambda'^2$, $s_i^{(0)} \rightarrow s_i^{(1)}$ in Eq. (10), we get representations of the type (10) for the first and second factors in the right member of Eq. (7). Substituting in Eq. (7) all the relations of the type (10) obtained in this way, and then setting $s_i^{(1)} = s_i$ in the resulting equation and cancelling a common factor, we find a functional equation for the function $\widetilde{\square}_1$. Introducing the dimensionless variables $x = p^2/\lambda^2$, $y = m^2/\lambda^2$, $t = \lambda'^2/\lambda^2$, we get in the usual way a Lee differential equation of the following form:

$$\frac{\partial}{\partial x}\widetilde{\Box}_1\left(\frac{x-y}{1-y},\, s_i,\, e^2,\, h_\alpha\right)$$
$$=\frac{1}{x-y}\widetilde{\Box}_1\left(\frac{x-y}{1-y}s_i,\, e^2,\, h_\alpha\right)$$
$$\times F\left(s_i, e^2, h_\alpha\varphi\left(\frac{x-y}{1-y},\, s_i,\, e^2,\, h_\alpha\right)\right),$$
$$F(s_i, e^2, h_\alpha)=\left[\frac{\partial}{\partial\xi}\Box_1(\xi,\, s_i, e^2,\, h_\alpha)\right]_{\xi=1},$$
$$h_\alpha\varphi=h_\alpha d_M^2\left(\frac{x-y}{1-y}, e^2\right)\widetilde{\Box}_1\left(\frac{x-y}{1-y},\, s_i, e^2,\, h_\alpha\right). \tag{11}$$

In a similar way one derives an equation for the second invariant charge in the infrared region for the four-vertex function:

$$\frac{\partial}{\partial x}h_\alpha\varphi\left(\frac{x-y}{1-y},\, s_i,\, e^2,\, h_\alpha\right)=\frac{1}{x-y}h_\alpha\varphi\left(\frac{x-y}{1-y},\, s_i,\, e^2,\, h_\alpha\right)$$
$$\times\Phi\left(s_i, e^2, h_\alpha\varphi\left(\frac{x-y}{1-y},\, s_i,\, e^2,\, h_\alpha\right)\right),$$
$$\Phi(s_i, e^2, h_\alpha)=\left[\frac{\partial}{\partial\xi}\varphi(\xi,\, s_i,\, e^2,\, h_\alpha)\right]_{\xi=1}. \tag{12}$$

Thus to find the infrared asymptotic behavior of the four-vertex function it is necessary to solve the equations (11) and (12) simultaneously.

3. Let us first solve Eq. (12) by using for Φ the expression obtained in low orders of perturbation theory. We get

$$h_\alpha\varphi\left(\frac{x-y}{1-y},\, s_i,\, e^2,\, h_\alpha\right)=\frac{1}{\beta}\left(\left|\frac{x-y}{1-y}\right|^{e^2\beta}-a\right)(h_\alpha\beta+e^2f_2), \tag{13}$$
$$\beta=f_1(s_i)-1/4\pi^2, \qquad a=e^2f_2(s_i)/[h_\alpha\beta+e^2f_2(s_i)]. \tag{14}$$

Then taking F from the perturbation-theory calculations and substituting the value of $h_\alpha\varphi$ from Eq. (13), we get the solution of Eq. (11) in the form

$$\widetilde{\Box}_1\left(\frac{x-y}{1-y}, s_i,\, e^2,\, h_\alpha\right)=\frac{1}{1-a}\left[\left|\frac{x-y}{1-y}\right|^{e^2f_1(s_i)}-a\left|\frac{x-y}{1-y}\right|^{e^2/4\pi^2}\right]. \tag{15}$$

Thus the existence of an infrared singularity of the function $\Box_1$ is determined by the sign of the function $f_1(s_i)$, which is given by Eq. (2a).

For $f_1(s_i) < 0$ and in the region $x \sim y$ the function $\widetilde{\Box}_1$ takes the form

$$\widetilde{\Box}_1\left(\frac{x-y}{1-y},\, s_i,\, e^2,\, h_\alpha\right)\approx\frac{1}{1-a}\left|\frac{x-y}{1-y}\right|^{-e^2|f_1(s_i)|}. \tag{16}$$

4. So far we have been considering the asymptotic behavior of four-vertex diagrams in the infrared region. A matter of much interest is the procedure for removing the infrared divergence from the probability for the meson-meson scattering process. It is well known that in low orders of perturbation theory after the removal of the infrared divergence one gets an incorrect dependence of the scattering cross section on the maximum energy of the soft photon that is emitted.[8] The simplest way to get the correct result is to apply the method of the renormalization group to sum the probabilities of processes with emission of arbitrary numbers of long-wavelength photons (cf. reference 4). Here, since in scalar electrodynamics the renormalization group is a two-charge group, and the second invariant charge has an infrared singularity [cf. Eq. (13)], the question arises as to how one is to remove the divergence that can be introduced into the probability of the process on account of the second invariant charge.

Let us first examine the procedure for removing the infrared catastrophe in low orders of perturbation theory. For this we shall use the method of generalized diagrams for the probabilities of the process that has been suggested by Abrikosov.[1] In the present case we must consider three separate sums of generalized diagrams, each of which does not contain an infrared divergence. All three sets are shown in Figs. 2, 3, and 4. The diagrams of Figs. 2a, 3a, and 4a represent the zeroth approximation of the scattering cross section for two mesons. The diagrams of Figs. 2b, 3b, and 4b represent the next approximation with respect to e^2 for the pure elastic scattering cross section. Finally, Figs. 2c, 3c, and 4c give the lowest approximation for the process of meson-meson scattering with emission of a soft photon with energy not exceeding $\omega_{\max}$.

In calculating the contribution of each diagram we must take into account the possible interchanges of the initial and final momenta of the particles. The diagrams of type c have not all been drawn, since they are very numerous. The diagrams shown represent the mutual scattering of particles of like sign. The total probability of the process of meson-meson scattering with the emission of an

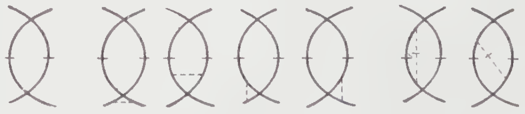
a b c

FIG. 2

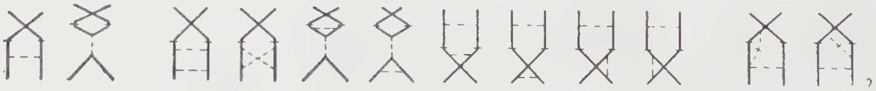
a b c

FIG. 3

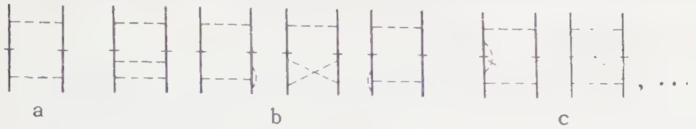

FIG. 4

arbitrary number of soft photons can be written in the form

$$M(\omega,\ s_i,\ e^2,\ h) = h^2 M_1(\omega,\ s_i,\ e^2,\ h) + he^2 M_2(\omega,\ s_i,\ e^2,\ h) + e^4 M_3(\omega,\ s_i,\ e^2,\ h), \tag{17}$$

where the first terms of the expansions of M_1, M_2, and M_3 are given respectively by the diagrams of Figs. 2, 3, and 4.

Since the emission of soft quanta is possible only owing to electromagnetic interactions, the breaking up of the probability into three terms as shown in Eq. (17) is entirely unambiguous. For example, a term $\sim h^2 e^2 \ln(\omega/m)$ can formally be assigned to either the first or the second term in Eq. (17). But if in the first term it gives a physically reasonable correction to the zeroth approximation for the process, proportional to $e^2 \ln(\omega/m)$, in the second term it would give a correction $\sim h \ln(\omega/m)$, which makes no sense, since the emission of a soft quantum cannot occur as a consequence of the four-boson interaction. We now note that the expansion of each term M_i in a power series in the interaction constants begins with unity, that is, they are all of the same order and can be separately normalized to unity.

We now assume that the constants e^2 and h satisfy the requirement

$$e^2 \ll h \ll 1. \tag{18}$$

In this approximation the second and third terms in Eq. (17) are small in comparison with the first, and we can drop them. Following the work of Blank[4] we get for the function $M_1(\omega,\ s_i,\ e^2,\ h)$ the group equation

$$M_1(\omega,\ s_i,\ e_2^2,\ h_2) = z_4^{-2} z^2 M_1(\omega,\ s_i,\ e_1^2,\ h_1). \tag{19}$$

Introducing normalized functions and repeating the arguments of Sec. 2, we get the following Lee differential equation:

$$\partial \widetilde{M}_1(x,\ s_i,\ e^2)/\partial x = x^{-1} \widetilde{M}_1(x,\ s_i,\ e^2)\, \mathfrak{M}_1(s_i,\ e^2), \tag{20}$$

where

$$\mathfrak{M}_1(s_i,\ e^2) = \left[\frac{\partial}{\partial \xi} \widetilde{M}_1(\xi,\ s_i,\ e^2)\right]_{\xi=1},\quad x = \frac{\omega}{\omega_0},\quad \widetilde{M}_1(1,\ s_i,\ e^2) = 1.$$

There is no dependence on the second charge h, since the lowest approximations $\mathfrak{M}_1$, which are used to get the main term of the renormalization-invariant expansion $\widetilde{M}_1(x,\ s_i,\ e^2,\ h)$ do not contain the constant of the four-boson interaction. The first two terms of the expansion $h^2 \widetilde{M}_1$ are calculated from the diagrams of Fig. 2a, b, c and are of the form

$$h^2 \widetilde{M}_1^{(2)}(x,\ s_i,\ e^2) = h^2 [1 + \varphi(s_1,\ s_2,\ s_3,\ e^2) \ln x]. \tag{21}$$

Here

$$\begin{aligned}\varphi(s_1,\ s_2,\ s_3,\ e^2) = \frac{e^2}{2\pi^2}\Bigg[1 &+ 4\frac{s_1 - 1/2}{\sqrt{s_1(s_1-1)}} \ln\frac{1+\sqrt{(s_1-1)/s_1}}{1-\sqrt{(s_1-1)/s_1}} \\ &+ 4\frac{1/2 + |s_2|}{\sqrt{|s_2|(1+|s_2|)}} \ln\frac{1+\sqrt{|s_2|/(1+|s_2|)}}{1-\sqrt{|s_2|/(1+|s_2|)}} \\ &+ 4\frac{1/2 + |s_3|}{\sqrt{|s_3|(1+|s_3|)}} \ln\frac{1+\sqrt{|s_3|/(1+|s_3|)}}{1-\sqrt{|s_3|/(1+|s_3|)}}\Bigg]\end{aligned} \tag{22}$$

for the scattering of particles of like signs.

Solving Eq. (20) by the use of Eq. (21), we get

$$\widetilde{M}_1(\omega/\omega_0,\ s_i,\ e^2) = \widetilde{M}_1(\omega_1/\omega_0,\ s_i,\ e^2) \exp[\varphi(s_i,\ e^2) \ln(\omega/\omega_1)]. \tag{23}$$

Thus one gets the physically correct dependence of the cross section on the maximum energy of the soft quanta that are emitted.

If we now replace Eq. (18) by a different condition, $h \ll e^2 \ll 1$, then in Eq. (17) the third term is the largest, and we can get for M_3 a result analogous to Eq. (23). The investigation of the intermediate case, for which $h \sim e^2 \ll 1$, encounters serious difficulties.

In conclusion the writer takes occasion to express his deep gratitude to D. V. Shirkov, under whose direction this work was done. The writer is also grateful to I. F. Ginzburg, L. P. Gor'kov, and L. D. Solov'ev for interesting discussions.

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SOVIET PHYSICS JETP VOLUME 12, NUMBER 2 FEBRUARY, 1961

EFFECT OF MULTIPLE SCATTERING ON THE PROPER FIELD OF A FAST CHARGED PARTICLE

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It is shown that due to the effect of multiple scattering the proper field of a fast charged particle passing through a medium decreases at a higher rate with increasing distance from the axis of motion than in the vacuum. This effect leads to the emission of radiation when the particle passes from one medium to another and affects the direct pair interaction.

1. Gol'dman[1] (see also reference 2) has shown that an ultrarelativistic electron entering a dense medium emits radiation with a spectrum which lies precisely in the frequency region of the quanta whose emission in the slowing down of a particle in an infinite medium is hindered by multiple scattering. Garibyan[3] established even earlier that the transition of a charged particle through the interface of two media with different dielectric properties is accompanied by the emission of radiation of a different type due to the readjustment of the proper field of the particle.

In the present paper it will be shown that the emission of the transition radiation predicted by the theory of Gol'dman is also connected with a readjustment of the proper field of the particle, which, however, is due to the effect of multiple scattering. This problem will be considered in Sec. 2, where we shall compute the number of quanta per unit interval of the spectrum which are equivalent to the proper field of the particle moving in the medium [formula (6)]. It will also be shown [formula (9)] that the number of equivalent photons in the vacuum, as determined by the well-known Weizsäcker-Williams formula,[4] agrees with good accuracy with the number of transition quanta added to the number of photons which are equivalent to the field of the particle in the medium.

At distances from the point of entry into the medium which are smaller than the cascade unit of length, where the transition quanta have not yet succeeded in being absorbed, the cross section for the direct creation of pairs will of course be proportional to the number of equivalent photons in the vacuum. Far away from the point of entry, at places where the transition radiation does not penetrate, the number of pairs created directly by the field of the particle will become proportional to the number of equivalent photons in the medium. A detailed review of this very reasonable situation will be given in Sec. 3.

Finally, in Sec. 4, we shall investigate the process of pair creation by a particle passing through a medium for the case where the distance at which the radiation field is formed is comparable with the cascade unit. Here it becomes impossible to separate out the cross section for the direct creation of pairs, and the total number of pairs at a depth much larger than the cascade unit is determined by formula (19).

2. To determine the number of photons equivalent to the proper field of the particle in the medium we obtain first an expression for the number of pairs in the first cascade unit created by a fast electron passing through the medium with energy $p_0 \gg 1$, and separate out the pairs which are created directly by the field of the particle. The number of the latter, as is well known,[4] is proportional to the number of photons connected with the proper field of the particle.

We shall start from the expression for the probability for bremsstrahlung per unit length of path in an infinite medium, derived by Migdal.[5,6] This expression has the following form for $k \ll p_0$:*

$$W_r(p_0,\ k) = \frac{e^2k}{2\pi^2}\int_0^\infty f(t)\,dt,$$

$$f(t) = \operatorname{Re} e^{i\omega t}\int (\xi\eta)\, w(\xi,\ t,\ \eta)\, d\xi d\eta, \tag{1}$$

where w is the Fourier component of the usual distribution function of the trajectories, $\omega = k/\sqrt{\epsilon}$ is the energy of a quantum with momentum k, $\epsilon = 1 - \omega_0^2/\omega^2$ is the dielectric constant of the medium, $\omega_0^2 = 4\pi nZe^2$, and n is the number of nuclei per unit volume.

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*We use a system of units in which $\hbar = m = c = 1$.

When the emitted bremsstrahlung quanta are absorbed they create pairs, which can, however, also be created directly by the field of the particle itself. To account for this effect it is sufficient to introduce under the integral sign in formula (1) the factor exp [− $n\sigma t/2$], which corrects for the absorption of quanta in the radiation process itself. The expression $n\sigma$ is the total cross section for the creation of pairs by a γ quantum with energy k per unit length of path in the medium. This procedure is equivalent to introducing an imaginary part of ϵ due to the absorption of γ quanta through pair creation. It can be shown that the corresponding change in the real part is negligibly small.

We thus obtain the following expression for the total number of pairs created per unit length of path in an infinite medium:

$$W_p(k, p_-)(n\sigma)^{-1}\widetilde{W}_r(p_0, k)\,dk\,dp_-;$$

$$\widetilde{W}_r = \frac{e^2k}{2\pi^2}\int_0^\infty e^{-n\sigma t/2}f(t)\,dt, \tag{2}$$

$$W_p(k, p_-)\,dp_- = \frac{1}{3kL}\left[1 + 2\,\frac{p_-^2 + (k-p_-)^2}{k^2}\right]dp_-. \tag{3}$$

Here $W_p(k, p_-)$ is the probability for pair creation by a γ quantum per unit length of path with an electron energy between p_- and $p_- + dp_-$, and L is the radiation length. If we denote the important time interval in the integral (1) by

$$\tau_k = \min\{2p_0^2/k, \sqrt{p_0^2L/2\pi k\cdot 137}\}$$

we can expand the factor exp [− $n\sigma\tau/2$] in formula (2) in a series, provided that $n\sigma\tau_k \ll 1$. Then Eq. (2) takes the form

$$\widetilde{W}_r \approx W_r + n\sigma q(k), \qquad q(k) = -\frac{e^2k}{4\pi^2}\int_0^\infty t\,dt f(t). \tag{4}$$

Comparing (2) and (4), we see that W_pq is the cross section for direct pair creation, which means that q(k)dk is the number of γ quanta in the spectral interval k, k + dk which are equivalent to the proper field of the particle.

Methods for calculating expressions of the type (2) or (4) have been discussed by various authors[1,2,5] These expressions have the feature that they contain logarithmic divergences for large radiation angles, because our method becomes invalid when the angular width of the radiation exceeds the average angle between the directions of emission of the particles of the pair. The latter is of the order of magnitude of k^{-1}. If we cut off the integration over angles at $\eta \sim k^{-1}$, we obtain

$$q(k) = \frac{e^2}{\pi k}\Bigl\{\ln\frac{\varkappa p_0^2}{k^2(1+\delta)} - 1 + \mathrm{Re}\int_0^\infty\left(\coth x - \frac{1}{x}\right)(1-\mu' x)\,e^{-\mu' x}\,dx\Bigr\}, \tag{5}$$

where $\kappa \approx 1$ and

$$\mu' = 2(1-i)s', \quad s' = s(1+\delta), \quad \delta = \omega_0^2p_0^2/k^2,$$

$$s \approx 1.37\cdot 10^3\sqrt{kL/p_0^2}.$$

Here k, p_0, and ω_0 are in mc^2 units, and L is in centimeters.

The integral in formula (5) reduces to zero when $s' \gg 1$ and is approximately equal to log s′ when $s' \ll 1$. For arbitrary values of s and δ, formula (5) can therefore be replaced by the simple approximate formula

$$q(k) \approx \frac{2e^2}{\pi k}\ln\left[\frac{\varkappa p_0}{k}\sqrt{\frac{s}{1+s(1+\delta)}}\right], \tag{6}$$

the inaccuracy of which shows up in the presence of the unknown factor $\kappa \sim 1$ in the argument of the logarithm. This factor will be omitted in the following.

As is known,[4] the number of photons in the spectral interval k, k + dk which are equivalent to the field of an electron moving in the vacuum with energy $p_0 \gg 1$ is given by $q_0(k)dk$ with

$$q_0(k) \approx \frac{2e^2}{\pi k}\ln\frac{\rho_{max}}{\rho_{min}} = \frac{2e^2}{\pi k}\ln\frac{p_0}{k}, \tag{7}$$

where $\rho_{min} \sim \lambda_c = \hbar/mc = 1$ and $\rho_{max} \sim p_0/k$ is the maximal distance from the axis of motion at which the Fourier component of the field of the particle is still appreciably different from zero. Comparing (6) and (7), we see that, due to the effects of the polarization of the medium and of multiple scattering, the field of the particle differs from zero only at distances of order $\rho'_{max} = (p_0/k)\sqrt{s/(1+s')}$. The same result can also be obtained directly through a study of the important values in the expression for the Fourier component of the field of the particle in the medium.

If the particle makes a transition from the medium to the vacuum, radiation should be emitted as a consequence of the readjustment of the field of the particle. It has been shown earlier[1,2] that the number of quanta of the transition radiation in the spectral interval k, k + dk can be written in the approximate form

$$N(k)\,dk \approx \frac{2e^2}{\pi k}\,dk\ln\frac{p_0}{k}\sqrt{\frac{1+s(1+\delta)}{s}}. \tag{8}$$

Comparing expressions (6) — (8), we find that

$$q_0(k) \approx N(k) + q(k). \tag{9}$$

This result has important consequences for the theory of the direct pair creation.

3. If the transition radiation occurs at distances of order $\tau_k \ll 1/n\sigma$, one can assume that at time t the quanta of the transition radiation create

$$W_p(k, p_-)\,N(k)\,\frac{1-e^{-n\sigma t}}{n\sigma}\,dk\,dp_-$$

pairs with total energy k, with electron energies between p_- and $p_- + dp_-$.

The number of pairs created directly by the field of the particle in time t is equal to

$$W_p(k, p_-)\,q(k)\,t\,dp_-\,dk.$$

For $t \ll 1/n\sigma$, the number of pairs created by the field of the particle is given by the expression

$$W_p[q+N]\,t\,dk\,dp_- \approx W_p q_0\,t\,dk\,dp_-. \tag{10}$$

If $t \gg 1/n\sigma$, the quanta of the transition radiation are completely absorbed, and the number of pairs created per unit length of path directly by the field of the particle will be equal to $q(k)W_p dk dp_-$, where q(k) is given by formula (6). If here $10^6 L \gg p_0 \gg 10^3\sqrt{L} \equiv p_1$ (L is measured in centimeters and p_0 in units of mc^2), the total cross section for the direct creation of pairs is given by the expression

$$\sigma_\pi = \frac{7}{9L137\pi}\ln\frac{p_0^2}{p_1}\ln p_1, \tag{11}$$

which is proportional only to the first power of the logarithm of the energy.

If $p_0 > 10^6 L$ and

$$\widetilde{s} \approx 1.37\cdot 10^3\,[kL/p_-(k-p_-)]^{1/2} \leqslant 1,$$

we must take account of the effect of multiple scattering on the pair creation by γ quanta.[6,7] Here the average angle between the directions of emission of the particles is widened and reaches the value $k^{-1}\widetilde{s}^{-1/2}$. In the integration over η in formula (4) we must therefore take $k^{-1}[(1+\widetilde{s})/\widetilde{s}]^{1/2}$ as the upper limit. We then find

$$q(k) \approx \frac{2e^2}{\pi k}\ln\left[\frac{p_\theta}{k}\sqrt{\frac{s\,(1+\widetilde{s})}{\widetilde{s}\,(1+s')}}\right] \approx \frac{e^2}{\pi k}\ln\frac{1+\widetilde{s}}{1+s'}. \tag{12}$$

Using the result of reference 7 for the cross section for direct pair creation per unit length of path in an infinite medium, we obtain the expression

$$d\sigma_\pi = \frac{\xi(\widetilde{s})\,dk\,dp_-}{3\pi\,137\,k^2L}\left[G(\widetilde{s}) + 2\,\frac{p_-^2+(k-p_-)^2}{k^2}\,\Phi(\widetilde{s})\right]\ln\frac{1+\widetilde{s}}{1+s'}, \tag{13}$$

where G and Φ are the functions introduced by Migdal,[5,6] and

$$\xi(\widetilde{s}) = \min\left\{2,\ \frac{\ln\left(190\,Z^{-1/3}\sqrt{(1+\widetilde{s})/\widetilde{s}}\right)}{\ln 190\,Z^{-1/3}}\right\}.$$

Formula (13) differs from the formula quoted in reference 7 by a factor in the argument of the logarithm which takes account of the effect of multiple scattering on the proper field of the particle. It describes the direct creation of pairs at large distances from the point of entry of the particle into the medium.

The results of reference 7 remain valid near the boundary. For the total number of pairs at an arbitrary depth t we find the expression*

$$\begin{aligned}&\frac{\xi(\widetilde{s})}{3kL}\left[G(\widetilde{s}) + 2\,\frac{p_-^2+(k-p_-)^2}{k^2}\,\Phi(\widetilde{s})\right]\\ &\times\left\{\frac{e^2}{\pi k}\left[t\ln\frac{p_0^2\,s\,(1+\widetilde{s})}{k^2\widetilde{s}\,(1+s')} + \frac{1-e^{-n\sigma t}}{n\sigma}\ln\frac{1+s'}{s}\right]\right.\\ &\left.+\frac{1}{n\sigma}\left[t - \frac{1-e^{-n\sigma t}}{n\sigma}\right]\frac{4\xi(s)}{3kL}\,\Phi(s)\right\}dk\,dp_-.\end{aligned} \tag{14}$$

4. All the results mentioned above are valid only if the absorption of the quanta in the process of radiation itself is small. If this condition is not fulfilled, expansion (4) becomes incorrect and it becomes generally impossible to separate out the bremsstrahlung cross section.† The inequality $n\sigma\tau_k \gg 1$ is equivalent to the condition $p_0^2/k \gg 10^{12}L$ (L in cm) and can be satisfied only for very soft quanta at reasonable electron energies. We shall therefore neglect in the following the effect of multiple scattering on the creation of pairs by γ quanta.

Carrying out the calculations, we find from formula (2)

$$\begin{aligned}\widetilde{W}_r = \frac{e}{16\pi k}\,\mathrm{Re}\left\{\left[\frac{n\sigma}{2} - \frac{ik(1+\delta)}{p_0^2}\right]\left[\frac{1}{\mu'+\nu} - \psi\left(1+\frac{\mu'+\nu}{2}\right)\right.\right.\\ \left.\left.+\ln\frac{\mu p_0^2}{4k^2}\right]\right\},\end{aligned} \tag{15}$$

where

$$\nu = 2(1+i)\,r, \qquad r \approx 4\cdot 10^{-8}\sqrt{p_0^2/kL},$$

and ψ is the logarithmic derivative of the γ function.

For $r \ll 1$ or $s' \gg r$ formula (15) reduces to (4), while for $r \gg 1$ or $s' \ll r$ (15) leads to

$$\widetilde{W}_r = n\sigma\,\frac{e^2}{\pi k}\ln\frac{p_0^2\,s}{k^2 r}. \tag{16}$$

If L is measured in centimeters and k in units of mc^2, we have $p_0^2 s/k^2 r \approx 10^{11}L/k$.

Under these conditions we must also change the expression for the number of transition quanta. For this purpose it is sufficient to introduce in the expression for ϵ in formula (16) of the author's paper[2] an imaginary part due to the absorption of γ quanta through pair creation. We then obtain

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*The last term in the curly brackets describes the contribution from the bremsstrahlung quanta.

†This was first pointed out by Landau and Pomeranchuk.[8]

$$N = \frac{e^2}{\pi k} \operatorname{Re} \Big\{ 2(\mu' + \nu) \times \int_0^\infty \frac{dz}{1+z} \int_0^\infty \exp\left[-(\mu'+\nu)\tau - \mu z \tanh\tau\right] d\tau + \int_0^\infty \left(\coth\tau - \frac{1}{\tau}\right)\left[1-(\mu'+\nu)\tau\right] \times \exp\left[-(\mu'+\nu)\tau\right] d\tau - 2 - \ln\frac{\mu'+\nu}{\mu} \Big\}. \tag{17}$$

For $r \gg 1$ and $r \gg s'$ we find from (17)

$$N = \frac{e^2}{\pi k} \ln\frac{r}{s}, \tag{18}$$

and for the total number of pairs with energy k at the depth $t \gg 1/n\sigma$ we obtain the expression

$$n\sigma \frac{e^2}{\pi k}\left[t \ln\frac{p_0^2 s}{k^2 r} + \frac{1}{n\sigma}\ln\frac{r}{s}\right]. \tag{19}$$

The region of validity of expressions (16), (18), and (19) is determined by the inequalities $r \gg 1$ and $r \gg s'$. In order for these two conditions to be fulfilled simultaneously, we must have

$$\lambda_C\, p_0^2\, 100\, L \gg k \gg \omega_0^2\, L\, \lambda_C, \tag{20}$$

where $\lambda_C = \hbar/mc$, and p_0, k, and ω_0 are measured in the units mc^2. The inequality (20) can be fulfilled only if $p_0 \gg p_2 = 10\omega_0 L/\lambda_C$. For lead $p_2 = 7.5 \times 10^{12}$ ev, and for aluminum $p_2 = 1.6 \times 10^{14}$ ev.

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LIMITS OF STABILITY AND PROTON AND TWO-PROTON RADIOACTIVITY OF NEUTRON-DEFICIENT ISOTOPES OF LIGHT NUCLEI

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Application of isotopic invariance principles to light nuclei yields a very simple relation between the neutron and proton binding energies in distant mirror nuclei. This relation enables us to establish the limits of stability of neutron-deficient isotopes of light nuclei with respect to proton emission and to predict the existence and properties of approximately 90 such isotopes. Nuclei are indicated for which proton radioactivity or the very unique phenomenon of two-proton radioactivity can be observed. The chief properties of this interesting phenomenon are analyzed.

By using the principles of isotopic invariance we can readily show that the difference ΔE_{np} between the binding energy E_n of the Z-th neutron in a nucleus ${}_N M_Z^A$ and the binding energy E_p of the Z-th proton in the mirror nucleus ${}_Z M_N^A$ is determined by the relation

$$\Delta E_{np} = E_n({}_N M_Z^A) - E_p({}_Z M_N^A)$$
$$= [E_{\mathrm{Coul}}({}_Z M_N^A) - E_{\mathrm{Coul}}({}_{Z-1} M_N^{A-1})]$$
$$- [E_{\mathrm{Coul}}({}_N M_Z^A) - E_{\mathrm{Coul}}({}_N M_{Z-1}^{A-1})], \qquad (1)$$

in which the first two terms describe the change in the Coulomb energy when one proton is removed from the nucleus, while the last two terms take into account the corresponding change when one neutron is removed from the nucleus (owing to the reduction in the nuclear dimensions).

Accurate to about 1%, the value of δE_{np} should in general be independent of N and determined only by the value of Z, so that instead of the relation $\delta E_{np} \approx 1.2(Z-1)(Z+N-1)^{-1/3}$ which is expected at first glance, we obtain

$$\Delta E_{np} \approx \Delta E_0 = E_n({}_Z M_Z^{2Z})$$
$$- E_p({}_Z M_Z^{2Z}) \approx 1.2(Z-1)(2Z-1)^{-1/3}. \qquad (2)$$

It can be readily shown that another simple expression of the consequences of the principles of isotopic invariance is a relation that characterizes the difference of masses of the remote mirror nuclei

$${}_Z M_N^A - {}_N M_Z^A \approx (Z-N)\,\Delta M_0, \qquad (3)$$

where

$$\Delta M_0 = {}_{A/2+1/2} M_{A/2-1/2}^A - {}_{A/2-1/2} M_{A/2+1/2}^A, \quad \text{for odd A,}$$
$$\Delta M_0 = \frac{1}{2}\left\{ {}_{A/2+1} M_{A/2-1}^A - {}_{A/2-1} M_{A/2+1}^A \right\} \quad \text{for even A.}$$

In many cases this formula may be useful and even more convenient than relation (2), which we use in this paper. As can be seen from Table I, relation (2) is confirmed by all the experimental data available for nuclei up to scandium ($Z = 21$).

Making use of (2), i.e., comparing δE_{np} with ΔE_0, or (if ΔE_1 is unknown) using the calculated value of the Coulomb energy, we can predict the properties of a rather large number of neutron-deficient isotopes of light nuclei (thereby adding to the similar isotopes discussed in the papers of Baz'[1] and Zel'dovich[2]) from the known properties of the corresponding mirror neutron-rich nuclei. Among the particular properties are the binding energies of the neutrons and the protons, the mass defect, the half lives of β^+ decay and its mechanism, and the possibility of observing proton and two-proton radioactivity.

A summary of all these properties, for almost 100 presently unknown isotopes, but which are stable to the emission of protons, is contained in a detailed communication which is now in press.[3] We give here only a general illustration (Table II) of the stability limits of the neutron-deficient isotopes of light nuclei. The conclusions regarding the position of these limits do not change if corrections similar to those considered by Swamy and Green[4] are introduced. These corrections are due to the Coulomb exchange interaction of the protons.

Proton and two-proton radioactivity should be

TABLE I. Difference between the binding energy of the Z-th neutron in the ${}_{N}M_{Z}^{A}$ nucleus and the binding energy of the Z-th proton of the nucleus ${}_{Z}M_{N}^{A}$ (ΔE_{np}).

Z \ N		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	$1{,}2\frac{Z-1}{(2Z-1)^{1/3}}$
H	1																						
He	2	0,76	0,76																				
L	3		0,84	0,84	0,83																		1,42
Be	4		1,63	1,64	1,64	1,70	1,78																1,91
B	5			1,90	1,86	1,86	2,02																2,33
C	6			3,01	2,94	2,76	2,76	2,64															2,73
N	7					2,88	3,01	3,01	3,02	3,2													3,10
O	8						3,55	3,54	3,54	3,25													3,45
F	9						3,50	3,26	3,55	3,55	3,43												3,78
Ne	10								3,91	4,05	4,05	(5,55)											4,10
Na	11									(5,81)	4,30	4,30											4,40
Mg	12											4,86	4,86	5,00									4,70
Al	13											5,21	5,08	5,08									4,98
Si	14													5,61	5,61	(4,95)							5,26
P	15													(5,09)	5,76	5,75							5,54
S	16															6,25	6,25						5,80
Cl	17																5,98	5,88					6,05
Ar	18																	6,76	6,76				6,32
K	19																		6,88	6,88			6,56
Ca	20																			7,43	7,43	7,31	6,80
Sc	21																			6,63	6,74	6,74	7,04

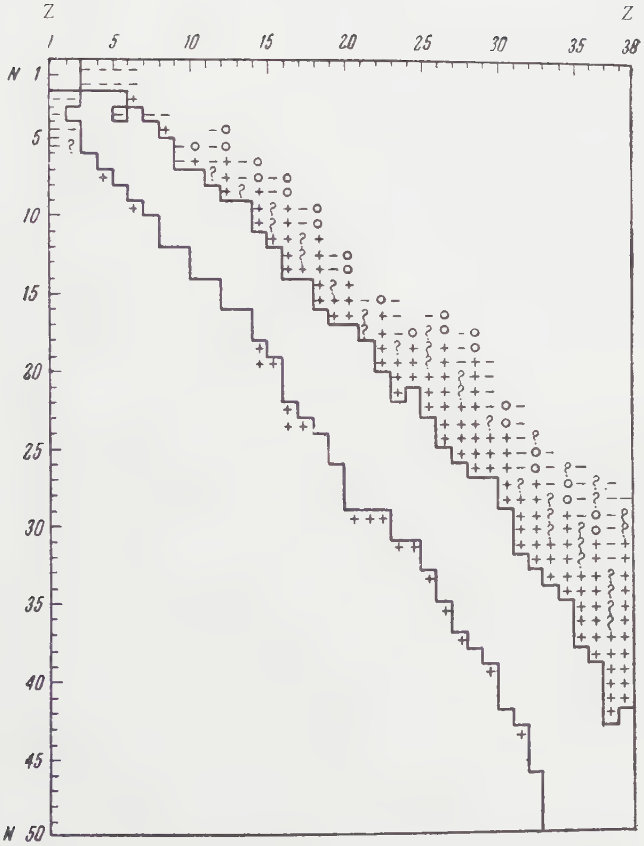

TABLE II. The continuous line outlines the area of the isotopes already known; + — predicted isotopes, stable to the emission of p and n, ? — doubtful p-stability, o — possible 2p-activity, — — isotopes known to be unstable to the emission of p or n.

observed near the stability limits of the neutron-deficient isotopes shown in Table II.

The probability of observation of the generally-trivial proton radioactivity is relatively small, for great difficulties are encountered in its experimental observation if the lifetimes of the p-decay are excessively short, while in the case of long p-decay time, this effect will be strongly screened by the β^+ decay. In the interval of observed p-decay times, from 10^{-12} sec (emulsion method) to 10 sec, the corresponding energies of the emitted protons range up to 0.04 Mev for Z = 10, 0.1 — 0.35 Mev for Z = 20, 0.2 — 0.7 Mev for Z = 30, and 0.35 — 1.1 Mev for Z = 40.

A much more interesting consequence of the considered properties of the neutron-deficient isotopes of light nuclei is the feasibility of two-proton radioactivity. The point is that for isotopes with even Z, instability to simultaneous emission of two protons can occur even when the binding energy of one proton is still positive; this takes place, for example, for Be^6 (reference 5). However, the presence of the Coulomb barrier, can cause such an instability to lead to two-proton radioactivity of many isotopes which are stable both to proton decay and to α decay.

Inasmuch as the width of the ground state of the Li^5 nucleus, which is formed in the decay $Be^6 \rightarrow Li^5 + p$, exceeds the instability of Be^6, we can consider in this case the emission of each of the two protons as independent.

On the other hand, if the energy that must be expended to detach one proton exceeds greatly the half width (reduced by the action of the Coulomb barrier) of the level from which the emission of the second proton takes place, we should have a very unique phenomenon of two-proton radioactivity — detachment of one proton is impossible, and correlated proton pairs are emitted.

One must not confuse this phenomenon with the usual chains of successive β^+ and (or) p decays of the type

$$\begin{tikzcd}
Zr^{69} \arrow[r, "p"] \arrow[d, "\beta+"'] & Y^{68} \arrow[r, "p"] \arrow[d, "\beta+"'] & Sr^{67} \arrow[r, "p"] \arrow[d, "\beta+"'] & Rb^{66} \arrow[r, "p"] \arrow[d, "\beta+"'] & Kr^{65} \arrow[r, "p"] \arrow[d, "\beta+"'] & Br^{64} \arrow[r, "p"] \arrow[d, "\beta+"'] & Se^{63} \arrow[d, "\beta+"'] \\
Y^{69} \arrow[r, "p"] \arrow[d, "\beta+"'] & Sr^{68} \arrow[r, "p"] \arrow[d, "\beta+"'] & Rb^{67} \arrow[r, "p"] \arrow[d, "\beta+"'] & Kr^{66} \arrow[d, "\beta+"'] \arrow[rrr, dashed, bend right, "2p"'] & & (Br^{65}) \arrow[r, "p"] \arrow[d, "\beta+"] & Se^{64}, \arrow[d, "\beta+"] \\
 & & & & & &
\end{tikzcd}$$

although even in chains of p-decay the possibility of two-protons emission can reduce substantially the lifetime of p-radioactive isotopes with even Z = 2m + 2, in those rare occasions when the energy of the 2p decay is

$$E_{pp}(2m+2 \to 2m) > 8E_p(2m+2 \to 2m+1).$$

Two-proton radioactivity, which is also conveniently observed with the aid of nuclear emulsions, can take place for Ne^{16}, $Mg^{17(18?)}$, $Si^{21(22?)}$, $S^{25(24?)}$, $Ar^{29(28?)}$, $Ca^{33(34?)}$, Ti^{38}, Cr^{42}, $Fe^{44(43?)}$, $Ni^{46(47?)}$, $Zn^{53(54?)}$, $Ge^{59(58?)}$, $Se^{63(62?)}$, $Kr^{67(66?)}$. The probability of the sub-barrier emission of two protons at the same time contains the product of two coefficients of penetrability of the barrier for protons, or else the product of the coefficient of penetrability for a doubly-charged particle and a small factor in front of the exponent, characterizing the probability of two-proton correlation in the nucleus. Consequently the lifetime of the isotopes relative to two-proton radioactive decay falls within limits that are convenient for registration in an energy interval which is considerably greater than that for proton radioactivity.

The possibility of emission of two protons in the case when the nucleus is stable to single-proton decay is a direct consequence of the excess of the binding energy of the even proton over the binding energy of the preceding odd proton, i.e., it is a consequence of the pairing effects. It is therefore clear that two-proton radioactive decay should exhibit the properties of a diproton that is stable within the confines of the nucleus, and that the angle and energy distributions and correlations of the emitted protons are connected with the character of the paired interaction of the protons and the original nuclei. On the other hand, inasmuch as the diproton breaks up in a "tunnel" under the barrier, this break-up determines the effective height of the barrier and its penetrability.

The simplest approach to the development of a theory of two-proton decay is to introduce the product of two ordinary proton barrier factors, i.e., an exponential multiplier of the form

$$w(E) = \exp\left\{-\frac{2\pi(Z-2)e^2\sqrt{M}}{\hbar\sqrt{2E_{pp}}}\left[\frac{1}{\sqrt{x}}+\frac{1}{\sqrt{1-x}}\right]\right\},$$

where E_{pp} is the sum of the energies of the two protons (energy of the emitted diproton), x and (1 − x) are the fractions of the energy belonging to each of the protons, and M is the proton mass. It is easily seen that w(E) has a maximum when x = 0.5, i.e., when the energies of the two protons are equal. In this case the quantity in the exponent is the same as for the subbarrier emission of the diproton of energy E_{pp} as a whole.

Thus, the break-up of the diproton under the barrier does not change the penetrability of the barrier if we disregard any supplementary interactions (for example, Coulomb repulsion) between two protons. The probability $w_x(E)$ of such a two-proton decay, in which the one proton has an energy $(0.5 + \kappa)E_{pp}$ the other $(0.5 - \kappa)E_{pp}$, with $\kappa \ll 0.5$, is connected with $w_{max}(E) = w_0(E)$, by the relation

$$\frac{w_x(E)}{w_0(E)} \approx \exp\left\{-\frac{6\pi(Z-2)e^2\sqrt{M}}{\hbar\sqrt{E_{pp}}}\varkappa^2\right\}.$$

It is obvious that the energy correlation between the two protons, in the case of two-proton decay that leads to nearly equal proton energies, is very strong. This correlation (the closeness of the energies of the subbarrier protons) can be noticed even in the almost instantaneous 2p decay, so that the probability of observing a two-proton radioactivity, say in emulsions, is greatly increased.

The interesting question of two-proton radioactivity of neutron-deficient isotopes of even elements is undoubtedly worthy of a more detailed special examination.

The most realistic way of obtaining neutron-deficient isotopes of light nuclei is to bombard the lightest stable isotopes of the neighboring nuclei with protons or He^3 nuclei of near-threshold energy and, in particular, to use reactions induced by heavy ions. It must be borne in mind, however, that owing to the smallness of the energy of the Coulomb barrier, the 'boiling off' of the protons will not be suppressed, and consequently the cross sections for the production of neutron-deficient isotopes should prove to be small.

In conclusion we note that in the case of a neutron excess we can observe a phenomenon analogous to two-proton radioactivity. The excitation energy of nuclei with an even number of protons and a large excess of neutrons, at which the emission of two neutrons is possible, may prove to be lower than the threshold for the emission of a single neutron, owing to pairing effects. Therefore there may exist for such nuclei an interval of excitation energies corresponding to the emission of pairs of neutrons correlated in angle and in energy, without emission of single neutrons. Cases which are even more frequently realizable are those in which the strongly excited states of neutron-rich nuclei with an even number of neutrons can disintegrate, with emission of both a single neutron and a correlated neutron pair. If similar excited states occur after the preceding β decay, they can be detected by the coincidences of the delayed neutron pairs.

I take this opportunity to thank Ya. B. Zel'dovich for useful discussions.

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SOVIET PHYSICS JETP VOLUME 12, NUMBER 2 FEBRUARY, 1961

ON THERMOMAGNETIC EFFECTS IN SUPERCONDUCTORS

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It is shown that the Righi-Leduc coefficient does not change when a metal changes from the normal to the superconducting state.

WE shall write down a transport equation for the distribution function for the electronic excitations to study thermomagnetic effects in superconductors.

Changing in the Hamiltonian for a system of electrons in a magnetic field from the second-quantization amplitudes to the amplitudes of the electronic excitations, following Bogolyubov,[1] we find an expression for the Lorentz force acting upon an excitation

$$F = \frac{e}{c}\,[\mathbf{v}\times\mathbf{H}]\,\frac{\xi}{|\xi|},$$

where

$$\mathbf{v} = \partial\varepsilon/\partial\mathbf{p}, \qquad \varepsilon = \sqrt{\xi^2+\Delta^2}, \qquad \xi = (p^2 - p_0^2)/2m$$

(ξ is the energy of an electron in the normal metal calculated from the Fermi surface, and Δ is the gap in the energy spectrum). The transport equation for electronic excitations, when there is a temperature gradient along the x axis and a magnetic field perpendicular to the thermal current, is then of the form

$$-\frac{\partial f}{\partial\varepsilon}\frac{\varepsilon}{T}v_x\frac{\partial T}{\partial x} + \frac{eH}{c}\left(v_y\frac{\partial f}{\partial p_x} - v_x\frac{\partial f}{\partial p_y}\right)\frac{\xi}{|\xi|} = -\frac{f-f_0}{\tau}. \qquad (1)$$

It was shown in reference 2 that the relaxation time τ is equal to $\tau = \tau_0\epsilon/|\xi|$, where τ_0 is the relaxation time for the normal electrons. We assume that either the dimensions of the solid are less than the penetration depth, or that $\partial H/\partial z = 0$ [in the latter case one must average Eq. (3) given in the following over z].

Solving Eq. (1) by the method of successive approximations ($f = f_0 + f^{(1)} + f^{(2)}$ we find the additional terms in the distribution function which are caused by the presence of the temperature gradient and the magnetic field:

$$f^{(1)} = \frac{p_x}{m}\tau_0\frac{\partial f_0}{\partial\varepsilon}\frac{\varepsilon}{T}\frac{\partial T}{\partial x}\frac{\xi}{|\xi|}, \qquad f^{(2)} = \tau_0^2\frac{1}{T}\frac{eH}{cm}\frac{\partial T}{\partial x}\frac{\varepsilon^2}{|\xi|}\frac{\partial f_0}{\partial\varepsilon}v_y,$$

$$f_0 = (e^{\varepsilon/kT}+1)^{-1}. \qquad (2)$$

The magnitude of the Righi-Leduc effect, which is well known to be the appearance of a temperature gradient perpendicular to the direction of the resulting heat current, is determined by the coefficient

$$L = \frac{\partial T}{\partial y'}\Big/\frac{\partial T}{\partial x'}H$$

(the x′ axis coincides with the direction of the resulting heat current). One can show easily that $L = Q_y/Q_xH$, where

$$Q_x = 2h^{-3}\int\varepsilon v_x f^{(1)}\,d\mathbf{p}; \quad Q_y = 2h^{-3}\int\varepsilon v_y f^{(2)}\,d\mathbf{p}.$$

From (2) we find

$$Q_y = 2h^{-3}\tau_0{}^2\frac{1}{T}\frac{eH}{mc}\frac{\partial T}{\partial x}\int\frac{\varepsilon^3}{|\xi|}\frac{\partial f_0}{\partial\varepsilon}v_y{}^2\,d\mathbf{p},$$

$$Q_x = 2h^{-3}\tau_0\frac{1}{T}\frac{\partial T}{\partial x}\int\frac{\varepsilon^3}{|\xi|}\frac{\partial f_0}{\partial\varepsilon}v_x{}^2\,d\mathbf{p}.$$

We have thus

$$Q_y/Q_x = \tau_0 eH/mc. \qquad (3)$$

This relation is independent of the magnitude Δ of the gap. The magnitude of the Righi-Leduc coefficient is thus not changed when the metal changes from the normal to the superconducting state and is equal to $L = \tau_0 e/mc$.

The Nernst-Ettingshausen effect which is the occurrence of an electric field perpendicular to the direction of the resulting heat current is clearly absent in the case of superconductors.

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SOVIET PHYSICS JETP VOLUME 12, NUMBER 2 FEBRUARY, 1961

Letters to the Editor

A FUNDAMENTAL METHOD FOR MEASURING TIME

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According to presently available methods, time is measured by the frequency of any given periodic process. The accuracy of the time measurement is determined here by the stability and the accuracy of the measurement of the frequency. The most accurate types of "frequency" methods for measuring time involve so-called atomic and molecular clocks, which use the frequency of the electromagnetic radiation generated during transition from one energy level to the next. The exceptionally high stability of these clocks is due to the fact that changes in external conditions have relatively little effect on the atomic or molecular energy spectra. However, the accuracy of these clocks is limited in principle and is determined by the characteristic width of the spectrum lines and hence by the lifetimes of the energy levels. The time intervals of the electromagnetic transitions in atoms and molecules which can be recorded as continuous electromagnetic radiation (not as particles, or γ quanta) range from 10^{-10} to 10^{-6} sec. These figures also determine the accuracy limit attained up to now in the measurement of time.

Another method for measuring time, different in principle from the "frequency" methods, is described below. By applying this method it is possible to increase considerably the accuracy of measurement. The principle of this method, which we shall call "nuclear," is very simple and is based on utilizing the number of disintegrations of radioactive nuclei (or unstable particles).

The components of the proposed method for measuring time are essentially as follows: a) a source — radioactive nuclei (or unstable particles) b) a transducer to record the particles or decay products, c) an electronic scaler and d) the display device — a counter calibrated directly in units of time corresponding to a given number of decay particles.

The stability of this "nuclear" clock exceeds the stability of atomic and molecular clocks because changes of external conditions have practically no influence on the decay constant of the nuclear transition.[1] At any rate the changes possible in this case would be far less than the corresponding changes in the energy spectra of the atoms or molecules.

Let us examine the basic limits in the proposed method for measuring time. This method is based on the uniformity of the decay process as a function of time, which in turn enables us to take into account the decay probability per unit of time. However, as has been shown previously, according to the general premises of the quantum theory, the law of decay is in principle not homogeneous in time (the decay law deviates from exponential) and hence the concept of decay probability per unit of time is in general only an approximation.[2] It is precisely this deviation of the decay law as a function of time which sets the basic limitations for the accuracy of the measurement of time in the proposed "nuclear" method. However, as has been shown previously, the condition under which the law of decay will be homogeneous in the time (i.e., with small non-exponential corrections), is

$$(\Gamma t / \hbar)\, e^{-\Gamma t/\hbar} \gg (\Gamma / E_0)^2, \qquad (1)$$

where Γ is the width of the decaying energy level and E_0 is its energy. It appears from (1) that the decay law becomes more homogeneous in time as Γ/E_0 decreases, i.e., the smaller the value of Γ and the larger E_0. We therefore conclude that for the purpose of measuring time it is desirable to select a decay state with small Γ, i.e., with a long lifetime. On the other hand, the selection of a long lifetime is justified because no corrections are needed to compensate for the decay of the source.* Let us make some elementary estimates on the basis of (1) in order to ascertain the fundamental limits of the proposed "nuclear" method for time measurement. For instance if the α emitter U^{238} with $T_{1/2} = 4.5 \times 10^9$ years is used as the radioactive nucleus and if the energy level of the α particles is $E_0 \approx 4$ Mev, then the (nonexponential) correction factor for a measured time interval of ~ 1000 years will be of the order of 10^{-60}. Thus, though in principle the "nuclear" method for measuring time also has an upper limit of accuracy, this limit can be moved very far away by a choice of nuclides with long lifetimes (small Γ) and high values of the energy E_0.

If the decay law is considered homogeneous in time, and has been shown to be true with a very high degree of accuracy (practically "infinite"), then the degree of accuracy of measuring time by the proposed "nuclear" method will be limited only by the statistical accuracy, i.e., the value of $N^{-1/2}$, where N is the number of disintegrations recorded by the counter. Thus a specified high accuracy in measuring time can be attained by selecting a sufficiently large N. This evidently reduces in turn to the choice of a high source power (number of disintegrations) and a low-inertia transducer capable of recording (without being "swamped") all the disintegration events. In order to obtain some practical figures let us consider the following examples: 1 g-atom of Li^8 is needed to measure 0.1 sec with an accuracy of 10^{-11}, ~ 1 g-atom of C^{11} is required to measure 1 min with an accuracy of 10^{-11}, and to measure 1 year with a degree of accuracy of 10^{-11}, either $\sim 10^8$ g-atoms of U^{238} or 10 g-atoms of Ni^{63} are required. To achieve 100% registration of all the disintegration events, one can use either a large number of ordinary (inertial) transducers or low-inertia ones recording the particles through stimulation of very short-lived nuclear energy levels.

It is evident that the accuracy of the proposed "nuclear" clock increases with increasing time interval to be measured, i.e., with increasing clock operating time.

Even though various technical problems may be encountered in the course of developing the proposed "nuclear" method for measuring time (such as the production of thin films of very large amounts of radioactive substance, the need to eliminate "collective effects," etc), it would be natural to attempt it experimentally, because the method enables us in principle to obtain a sharp increase in the accuracy of time measurement, in comparison with even the most highly developed contemporary "frequency" methods.

Finally, let us note that the development of a very accurate method for measuring time will permit a detailed investigation of the decay law of short-lived physical systems (especially the deviation of the decay law from the exponential), particularly of elementary unstable particles, thus yielding valuable information on elementary-particle interactions.[2]

In conclusion, I express my gratitude to Academician I. E. Tamm, Professor V. L. Ginzburg, Professor E. L. Feĭnberg, and to all the participants in the theoretical seminar at the Physics Institute of the Academy of Sciences, and also to Professor G. I. Petrashen, Professor S. É. Khaĭkin, Yu. N. Demkov, and A. M. Khalfin for valuable discussions and comments.

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[*] It must be noted, to be sure, that it is not absolutely necessary to use very long-lived states for the proposed method of measuring time, because corrections for decay can be obtained by determining the lifetime by means of an experiment on resonance scattering of decay products, which is independent of the experiment on the time measurement. Let us also note that the choice of very long-lived states is not convenient for practical purposes, because it will involve a very large amount of radioactive substance.

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DETERMINATION OF THE CHARGE-EXCHANGE CROSS SECTION FOR PION-PION COLLISION FROM THE ANALYSIS OF THE $\pi^- + p \rightarrow \pi^- + \pi^+ + n$ REACTION AT 290 Mev

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THE importance of studying π-π interactions is evident and has been reiterated earlier.[1,2] However, the experimental determination of π-π scattering data is very difficult because one has to use only indirect methods. All the available information on π-π interaction is given in Table I. It should be pointed out that only order-of-magnitude measurements of the π-π interaction cross section were attempted in references 1, 2, and 3.

In the present paper use was made of photoemulsion data (200 events) obtained from the study of the reaction

$$\pi^- + p \rightarrow \pi^- + \pi^+ + n \qquad (1)$$

TABLE I. Data on π-π interaction cross sections

Process	$\sigma_{\pi\pi}$, mb	Initial reaction
$\pi^+ + \pi^+ \to \pi^+ + \pi^+$	~ 250 [7] *	$\tau^+ \to \pi^+ + \pi^+ + \pi^-$
$\pi^+ + \pi^- \to \pi^+ + \pi^-$	~ 20 [8] **	$\pi^- + p \to \pi^+ + \pi^- + n$ (960 Mev)
$\pi^- + \pi^0 \to \pi^- + \pi^0$	~ 40 [8] **	$\pi^- + p \to \pi^- + \pi^0 + p$ (960 Mev)
$\pi^- + \pi^0 \to \pi^- + \pi^0$	~ 30 [9] **	$\pi^- + p \to \pi^- + \pi^0 + p$ (1 Bev)
$\pi^+ + \pi^- \to \pi^0 + \pi^0$	$\left(4^{+6}_{-4}\right)$ *,***	$\pi^- + p \to \pi^+ + \pi^- + n$ (290 Mev)

* At zero meson energy.
** Averaged over meson energy.
*** Present paper.

The emulsion stacks were irradiated in the negative-pion beam of the synchrocyclotron of the Laboratory of Nuclear Problems (Joint Institute for Nuclear Research). The average energy of the primary pions was found to be 290 ± 15 Mev, after correction for the slowing down in the emulsion. The preliminary results on the energy and angular distributions of the secondary particles from reaction (1) were reported at the Kiev Conference on High Energy Physics in July, 1959, and published.[3]

As a theoretical basis for the reduction of the experimental data we used the results of A. A. Ansel'm and V. N. Gribov[4] in which it is shown that the amplitude of the charge-exchange process $\pi^- + \pi^+ \longrightarrow \pi^0 + \pi^0$ at zero energy can be determined from the energy distribution of secondary particles in a reaction such as (1) near threshold. The inclusion of the interaction of particles in the final state, under the assumption that the interaction is non-resonant, permits one to write for the energy distribution of the secondary particles (correct to terms linear in kr_0, where k is the meson momentum and is of the same order as the total kinetic energy of the three particles in the c.m.s. and r_0 is the reaction radius)

$$d\sigma / d\Gamma = A(1 + ck_{12} + dk_{13}). \tag{2}$$

Here A is a constant determined by the total cross section for reaction (1). The coefficients c and d are associated with the π-π and π-n charge-exchange amplitudes at zero energy, k_{12} and k_{13} are the absolute values of the relative π^+-π^- and π^+-n momenta, respectively; $d\Gamma$ is an element of phase volume. Making use of isotopic invariance one can write:

$$c/d = (a_2 - a_0)/\sqrt{3}(b_{1/2} - b_{3/2}), \tag{3}$$

where a_0 and a_2 are the amplitudes of pion-pion scattering at zero energy in states with isotopic spin 0 and 2, $b_{1/2}$ and $b_{3/2}$ are the amplitudes of pion-nucleon scattering at zero energy in states with isotopic spin ½ and ³⁄₂. Since the difference $(b_{1/2} - b_{3/2})$ is known,[5] one can find the difference of the amplitudes $a_2 - a_0$ and, hence, the exchange amplitude $a_{12} = \frac{1}{3}(a_2 - a_0)$ from the experimentally determined ratio c/d. However, this theory may not be quite applicable at 290 Mev. The point is that the theory is linear in kr_0, so that it is necessary that terms quadratic in kr_0 be small compared to the linear terms and, hence, that the P phase shifts of the π-n and π-π scattering be small. In our case the kinetic energy in the c.m.s. is about 90 Mev. The average energy of the particles in the final state is about 40 Mev. At this energy one of the P-phase shifts of π-n scattering, δ_{33}, is comparable with the S phases δ_3 and δ_1.

Nevertheless, we attempted to reduce the experimental data in an effort to find out how well Eq. (2) describes the experimental situation. With this aim, the entire kinematically available region in the k_{12}, k_{13} plane was divided into regions containing 13 points or more, the density of points within a region being approximately constant. Thus, the whole region was divided into 9 parts. Owing to the energy dispersion (±15 Mev), only those events were chosen which were situated in a region internal to the limiting values (see Fig. 1). If Eq. (2) is correct, the points having the coordinates x_i, y_i, z_i should lie on a plane. Here x_i and y_i are approximately equal to the average values $\overline{k_{13}}$ and $\overline{k_{12}}$ for each region and z_i is equal

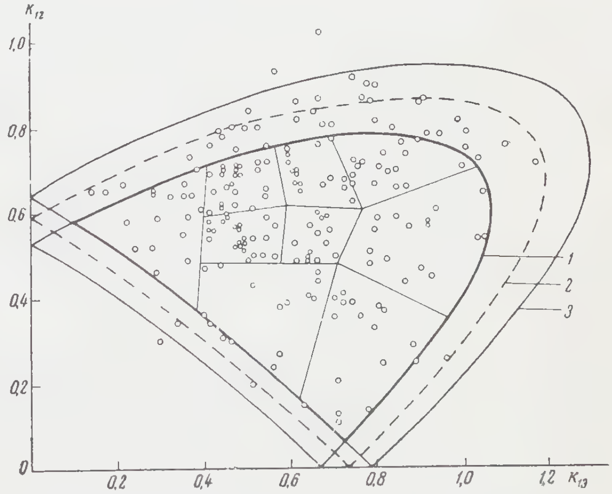

FIG. 1. Distribution of events for $\pi^- + p \to \pi^- + \pi^+ + n$ reaction at 290 ± 15 Mev in the k_{12}, k_{13} plane. Here k_{12}, k_{13} are the relative momenta $(\pi^+ - \pi^-)$ and $(\pi^+ - n)$ in units of $\mu_\pi c$. Curves 1, 2, 3 restrict the kinematically available regions for energies of 275, 290, 305 Mev, respectively.

to the total number of points in each region divided by the average value of the phase volume. The analysis carried out with a χ^2 distribution[6] has shown that within the experimental errors the points lie on a plane. The ratio c/d as obtained from the equation of the plane is equal to $-(0.76 \pm 0.65)$. It should be noted that other divisions analogous to that shown in Fig. 1 lead to the same value of c/d. From Eq. (3) one obtains a value $-(5 \pm 4) \times 10^{-14}$ cm for the difference $(a_2 - a_0)$; this corresponds to a charge-exchange cross section $\sigma_{\pi^+-\pi^-\to\pi^0+\pi^0} = 4\pi a_{12}^2 = 4^{+6}_{-4}$ mb. The given experimental errors are determined by statistics and not by inaccuracies of the theory at our energy which have not been included.

All the data available at present concerning the amplitude of S-wave π-π scattering are listed in Table II. As is seen from the table, the results of various authors differ both in absolute value and in sign. In such a situation it is of great interest to obtain more accurate data. Since in our treatment the accuracy of the theory is the main problem, it is more useful to run an experiment at lower energies, where the theoretical assumptions are more valid. At present such an experiment is being performed at an energy of 240 — 250 Mev (40 — 50 Mev in the c.m.s.).

TABLE II. S wave π-π scattering lengths ($\hbar/\mu_\pi c$ units)

a_2	a_0	$a_2 - a$	Initial reaction
~ 1			$\tau^+ \to \pi^+ + \pi^+ + \pi^-$ [7]
		$-(0,35 \pm 0,30)$	$\pi^- + p \to \pi^+ + \pi^- + n$*
	~ 1		$\pi + N \to \pi + N$ [10]
$-0,48$	$-0,8$	$0,3$	$K^\pm \to 3\pi$ [11]
$-0,3$	-1	$0,7$	$K^\pm \to 3\pi$ [12]
	~ 1		$\pi + N \to \pi + N$ [13]

*Present paper.

The authors are grateful to Prof. V. P. Dzhelepov for his interest and attention and also to A. A. Ansel'm and V. N. Gribov for valuable discussions and advice and to S. N. Sokolov for help in the data reduction.

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THE NATURE OF THE PARTICLE BEAMS IN THE CORE OF EXTENSIVE AIR SHOWERS

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In a previous article,[1] the existence of a peculiar feature in the lateral distribution of shower particles in the core region of extensive air showers (EAS) was reported. Narrow beams consisting of a large number of particles (from four to fifteen) were detected in studying the core structure by means of a diffusion chamber. The experimental results obtained previously made it possible to regard the observed particle beams either as the cores of electron-photon showers produced by π^0 mesons, or as groups of high-energy μ mesons. It will be shown that the second hypothesis is the more likely one.

Let us assume that the beams under consideration represent the cores of electron-photon showers initiated in the decay of π^0 mesons produced in nuclear interactions. We shall estimate the energy of the primary particle which produced the shower, and the altitude of shower production above the observation level. For this purpose, it is necessary that, in a circle with 4 cm radius (which corresponds to the maximum dimension of the observed particle beams), the average number of particles be $n = 7$ (which corresponds to the average number of particles in the beams). From the results[2] of the electromagnetic cascade theory for the lateral structure of an electron-photon shower initiated by a particle with a finite energy E_0, we find that the required number of particles in a circle with the given radius can, at any stage of shower development, be produced only for $E_0 \geq 10^{12}$ ev. Moreover, at $E_0 = 10^{12}$ ev, the maximum number of particles in a circle with the given radius is attained at the depth of $\sim 2t$, and is equal to ~ 6, which is in agreement with the experiment.*

The energy spectrum of electrons and photons in a shower produced by a 10^{12} ev particle has, at the depth of 2t, the following form (where N_e and N_{ph} are the number of electrons and photons respectively with energy greater than E):

E	10^8	10^9	10^{10}	10^{11}
$N_e(>E)$	5.5	4.0	2.5	0.5
$N_{\mathbf{ph}}(>E)$	10	8.0	4.0	0.8

In order to study the shower-producing properties of the particles belonging to the bunches, a TF-1 glass plate with a high lead content was placed on top of the diffusion chamber in the present experiment. The plate was 0.8 t thick. (The critical energy in glass is $\beta = 14$ Mev, and the radiation length is $t_0 = 9\ \mathrm{g/cm^2}$.) According to the calculations of Arley,[3] the number of particles in the beam should, after traversing 0.8 t of lead glass, multiply by a factor of four. Moreover, one should consider only those particles under the glass having an energy above 10^8 ev, since the distance between the sensitive layer of the chamber and the glass amounts to 25 cm, and particles having lower energies may, because of a large deflection angle, merge with the background. Thus, beams with four or more particles should, after traversing the lead glass, contain on the average ≥ 15 particles.†

In the experiment, the lead glass covered half of the diffusion chamber. During 850 hours of operation, 16 particle beams with ≥ 4 particles were detected in the uncovered part of the chamber. The number of particles in the beams, their distance from the shower axis r, and the total number of particles in the shower to which the beam belongs N, are shown in the table. Measurements were made under the lead glass during 440 hours. The features of the beams observed under the lead glass are also presented in the table.‡

On the average, one should observe, during the above-mentioned period, 8 beams with ≥ 15 particles under the lead glass. In reality, no beams with > 8 particles were observed under the lead glass. In order to assess the probability of such an event, it is necessary to know the character of the fluctuations in the development of showers produced by electrons and photons in the lead glass. Calculations** show that in spite of the small thickness of matter, fluctuations do not play an important role, since many particles are present in the beam.

The diffusion chamber was placed above a two-row array of ionization chambers.[4] The first row of ionization chambers was shielded by a 5 t layer of lead, and was used for determining the energy flux of the electron-photon component. The second row of ionization chambers, placed under a composite lead and graphite absorber, was used for the determination of the energy flux of the nuclear-active component. Unfortunately, it was not possible to trace fully the transformation of a particle beam in the absorber, since it is accompanied by the electron-photon and nuclear-active components of EAS. The observed large multiplication during the passage of the beam through the first row of chambers may be due to the electron-photon component of EAS, since the area of an ionization chamber is roughly 20 times greater than the dimensions of the particle beam. Ionization bursts in the second row of chambers are also observed during the passage of the beam. Although a fraction of the bursts are possibly due to the nuclear-active component, the bursts ob-

Open part of the chamber			Part of the chamber under the lead glass			Open part of the chamber			Part of the chamber under the lead glass		
r, m	$N \cdot 10^{-3}$	n	r, m	$N \cdot 10^{-3}$	n	r, m	$N \cdot 10^{-3}$	n	r, m	$N \cdot 10^{-3}$	n
1	5	5	1	22	5	5	50	7	6	880	8
1.5	30	6	1	20	5	5	20	8	7	120	7
1.5	30	10	1	10	6	6	100	5	10	60	6
2	400	6	1	31	5	8	650	5	10	600	4
2	60	13	1.5	10	4	9	40	9	12	500	4
3	150	5	2	100	4	10	100	5			
4	200	4	2	350	8	11	1100	12			
5	50	5	2.5	630	6	12	200	4			

served in the second row of chambers characterize the passage of the beam through the absorber more directly than the bursts in the first row, since the flux of nuclear-active particles in the shower core amounts to about 1% of all charged particles. The experimental size distribution of bursts in the second row of chambers is as follows (where ΔE is the energy interval, and Φ_E is the observed number of events in the given interval††):

ΔE, ev	$<2\cdot 10^9$	$2\cdot 10^9$—$2\cdot 10^{10}$	$>2\cdot 10^{10}$
Φ_E	39	7	2
Φ_B	40	5	3

Let us assume that the observed particle beams consist of high-energy μ mesons. The main process leading to the appearance of bursts of the size given above is the production of electron-positron pairs by μ mesons. Assuming a μ-meson energy $E = 10^{13}$ ev, as found in an earlier estimate,[1] we find that the probability that a single μ meson produces a pair of $>10^9$ ev in the lead-graphite absorber (~ 10 t) is 0.3. This means that the passage of μ mesons should, in general, be accompanied by small bursts in the second row of ionization chambers. A comparison of the experimental distribution of bursts in the second row of ionization chambers with the theoretical distribution Φ_B, obtained taking pair production by μ mesons into account,[5] is given above.

If we assume that the particle beam consists of μ mesons, then the relative increase in the particle number during the passage through 0.8 t of lead glass due to electromagnetic interactions of μ mesons should be negligible. This is borne out by the experiment.

Thus, the absence of multiplication during the passage of the particle beam through a 0.8 t thickness of lead glass strengthens the hypothesis that the beam consists of high-energy μ mesons. Also, this assumption is not contradicted by the ionization chamber data.

In conclusion, the authors would like to thank L. G. Smolenskiĭ and B. A. Zelenov for help in carrying out the experiment, and S. F. Semenko for help with the calculations.

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*For $t_0 = 10^{11}$ ev, the maximum number of particles in the circle with the given radius is roughly 50 times smaller than for $E_0 \approx 10^{12}$ ev.

†The observed particle beams cannot be due to low-energy ($E_0 < 10^9$ ev) electrons and photons, since, in that case, the trajectories would not be collinear.

‡Although the difference in the frequency of beams in the open part of the chamber and under the lead glass lies within the limits of statistical error, it is possible that a certain increase in the beam frequency under the lead glass is due to the multiplication of single high-energy electrons and photons in the lead glass.

**We have carried out the calculations assuming a Furry distribution for the fluctuations (which leads to a large overestimate of events with a low multiplication). Even in this limiting case, we would be able to observe a picture under the lead glass for which the table predicts a probability of only $<10^{-2}$.

††The detection threshold is equal to ten relativistic particles, which corresponds to an energy of $\sim 10^9$ ev transferred to the electron-photon component.

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100

NON-MONOTONIC DEPENDENCE OF THE SURFACE IMPEDANCE OF TIN ON THE MAGNETIC FIELD AT 1.9 MEGACYCLES

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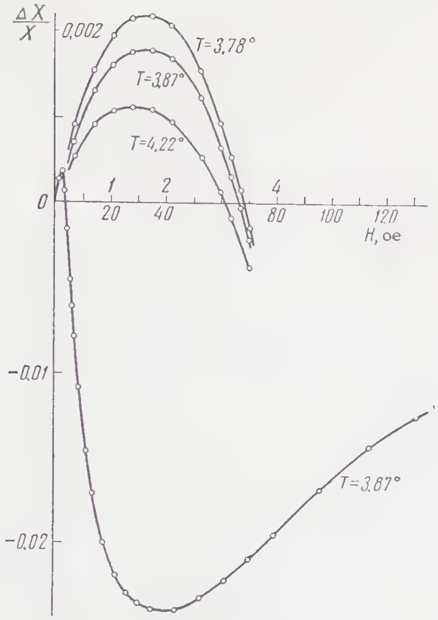
WE carried out preliminary experiments on measuring the reactive part of the surface impedance of Sn at 1.9 Mc/sec and helium temperatures. Samples of a cylindrical shape were placed in the coil of an oscillating circuit. On applying the magnetic field the frequency of the generator was changed by change of reactive component of the sample impedance. The frequency shift was measured to an accuracy of 0.01 cycles by means of apparatus, the detailed description of which will be published shortly.

In the diagram are plotted the results of measurement on one of the samples placed in a mag-



netic field which was parallel to the axis of the coil and the specimen. Along the ordinate axis is plotted the relative change of the reactive component of the impedance relative to the total magnitude, the latter being determined from the change in the oscillator frequency when the sample has passed into the superconducting state. In the top portion of the diagram are plotted the first parts of the curves at various temperatures on a larger scale.

The samples, to which the quoted results apply, consisted of some large crystals, whose orientations differed by 2 — 3°, and, obviously, were grown from one seed. The [001] axes of these crystals were at an angle of ~ 35° with the sample axis, and the angle between the [100] axis and the projection of axis of the sample on the (001) plane was ~ 30°. Similar results were obtained on another sample, the axis of which made an angle of ~ 70° with the [001] axis.

These two specimens were prepared from tin, containing approximately $< 10^{-4}\%$ impurities $[\rho(4.2°\mathrm{K})/\rho(20°\mathrm{K}) \approx 1 \times 10^{-5}]$ and cast in cylindrical quartz ampules, the ends of which were drawn out (sample diameter 8 mm, length of the cylindrical portion 40 mm, total length of the sample 60 mm.) The inner surface of the ampule was covered with a layer of carbon before casting. The sample was placed in the apparatus together with the ampule, which protected it from damage. After one of the samples had undergone a series of changes, it was removed from the ampule and the investigation repeated. In this case the character of the ΔX/X dependence was completely altered. Instead of a curve with two extrema, a monotonic reduction of X was observed in all the ranges of the field investigated. For a series of samples, prepared from some less pure tin and removed from the ampule before the measurements, the X-H dependence was also monotonically decreasing.

At present it is difficult to state any definite conclusions about the nature of the observed phenomena. A cyclotron-resonance experiment to explain its origin would require the introduction of a mean free path on the order of several centimeters. There is a striking similarity between our results and the oscillations of surface impedance at 9400 Mc/sec in a field of the order of several oersted, observed by M. S. Khaĭkin, for which there is still no explanation. A conjecture can be made that both the two effects are based on a mechanism which is independent of the frequency. To us it seems advisable to measure the impedance at still lower frequencies and make a static measurement of the magnetic susceptibility in the weak-field range.

We express our thanks to M. S. Khaĭkin for acquainting us with his results prior to their publication.

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101

CONCERNING CYCLOTRON RESONANCE IN TIN

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In order to use cyclotron resonance studies[1] to explain the features of the structure of the Fermi surface of a metal, it is necessary to develop methods for analyzing cyclotron resonance spectra, which sometimes contain several tens of minima of the metal surface resistance.[2] The Kaner and Azbel' investigation[1] of the relative depth of resonances as a function of the order of resonance is unreliable, primarily because the depth of resonances can only be determined approximately. This is clearly illustrated by the spectrum in Fig. 1, where the observable resonances partially overlap. However such indeterminacy occurs also in the more favorable spectrum of Fig. 2, and at the same time it is always made worse by the strong dependence of the depth of resonances on temperature and on the inclination of the magnetic field relative to the surface of the sample. A study of the depth of resonances as a function of the electromagnetic-field frequency[1] is also inadvisable, owing to the difficulty of varying the frequency substantially; a small frequency variation, which is relatively easy to realize, obviously cannot give precise results.

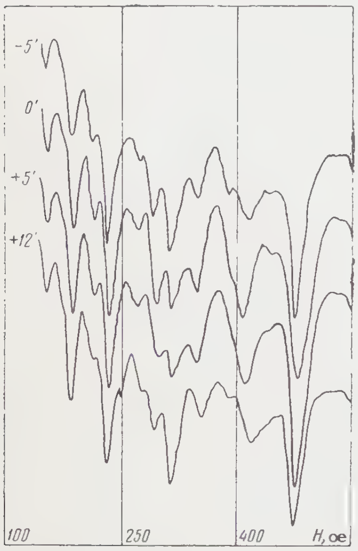

FIG. 1. The effect of inclination of a constant magnetic field to the surface face of a specimen on the character of cyclotron-resonance spectra. The ordinates represent the logarithmic derivative of the surface reactance of tin (in relative units). The angles of inclination of the field are indicated to the left of the curves, with an accuracy of ±1′. The temperature of the sample was 2.4 deg K.

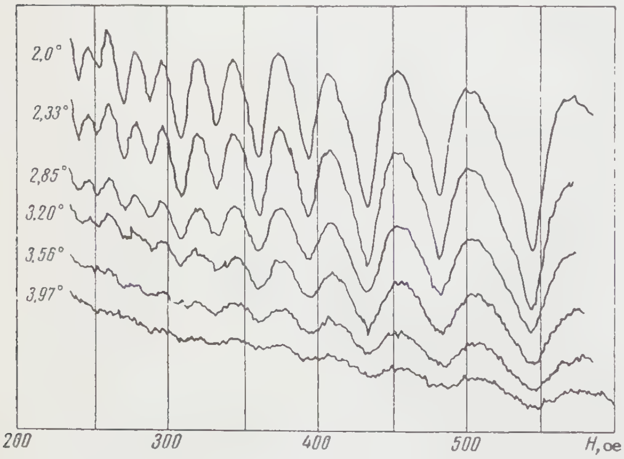

FIG. 2. The effect of the sample temperature on the nature of the cyclotron-resonance spectra. The temperature in deg K is indicated on the left of the curves. The magnetic field is directed along the bisector of the angle between the tetragonal and binary axes of the tin crystal.

Chambers[3] noted that the dependence of the depth of cyclotron resonances on the inclination of the magnetic field ought to be different for different groups of electrons: strong for electrons with velocity along the field, and weak for electrons which do not have such a velocity and move in closed (or almost closed) orbits.* Observation of this effect should obviously facilitate the analysis of cyclotron resonance data. Fawcett[5] has observed a different dependence of the depth of two broad minima in the surface resistance of aluminum on the field inclination, within the limits of ±4 deg.

Figure 1 presents the results of an experiment on the effect of a small inclination of the magnetic field to the surface faces of the sample on the character of the cyclotron resonance spectra. The measurements were carried out by the frequency-modulation method.[2,6] The sample was a single crystal of very pure tin with dimensions 13 × 6 × 1 mm. The tetragonal axis of the crystal was directed along the sample, and the binary axes were parallel to its two smaller dimensions. The high-frequency current flowed along the sample, and the high-frequency and constant magnetic fields were parallel to the binary axis in the face of the sample.[2]

As is apparent in Fig. 1, inclination of the magnetic field by a few minutes of angle affected strongly the depth of resonance of one sequence without affecting the five deeper and sharper resonances. These latter are related to the main sequence of resonances, found in reference 2 for a given field orientation, which correspond to an effective mass 0.27 m_e. The insensitivity to the inclination of the field indicates that these resonances are caused by electrons which do not have a significant velocity along the field direction, i.e., by electrons belonging to the central section of the Fermi surface by the (100) plane perpendicular to the constant magnetic field.

The other sequence of resonances, which disappear rapidly with increasing inclination of the field, is due to the group of electrons which have a significant velocity along the direction of the magnetic field. This group of electrons, the effective mass of which is equal to 0.49 m_e, should belong to a non-central extremal section of the Fermi surface by the (100) plane.

Experiments, the results of one of which are presented in Fig. 2, have shown that the depth of cyclotron resonance increases quickly with decreasing temperature of the sample, and that their width also decreases somewhat; for several other orientations of the field such a relation is even steeper than in Fig. 2. This indicates that the parameter $\omega\tau$ is not large enough in spite of the fact that the samples were made of very pure tin (an estimate yields $\omega\tau \sim 50$) in which the mean free path of the electrons, is possibly already limited by isotopic inhomogeneity of the metal.[7] Obviously a lowering of the sample temperature is necessary to resolve the resonances and to increase the accuracy of measurements of the resonant field values. However a more important result of these experiments is that, as seen from Fig. 2, the resonant field values do not depend on the temperature, at least, in regions below 3 deg K (an insignificant increase in the resonant fields values is noticed above 3 deg K). This fact permits us to assume that the values of the effective masses found from the cyclotron-resonance spectra actually correspond to definite sections of the Fermi surface. The insufficient value of the parameter $\omega\tau$ could lead to an averaging of the experimentally-obtained effective masses over a considerable region of the Fermi surface, thus greatly reducing the value of these data. A similar partial averaging apparently takes place at temperatures above 3 deg K.

It is appropriate to note the sharp increase in the number of observable resonances in each sequence with decreasing temperature, owing to the increase in the value of $\omega\tau$. This guarantees a substantial increase in the accuracy of measurements of the effective mass at lower sample temperatures.

The author thanks P. L. Kapitza and A. I. Shal'nikov for their interest in this work.

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INSTABILITY OF UNIFORM PRECESSION OF MAGNETIZATION

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IT is obvious that the occurrence of inhomogeneous magnetization in magnetic systems in a constant external magnetic field leads to a breakdown of the uniform precession. We shall consider therefore that the homogeneous precession is unstable if small fluctuations lead to growing magnetization waves. We shall investigate the stability of the uniform precession of a series of ferromagnetic and ferrimagnetic systems.

1. The change in the magnetization of a ferromagnet can be described by the equation

$$\dot{\mathbf{M}} = -\gamma[\mathbf{M}\times\mathbf{H}_{\text{eff}}],$$
$$\mathbf{H}_{\text{eff}} = \mathbf{H}_0 + \lambda\mathbf{M} + H_{ex}(l^2/M)\nabla^2\mathbf{M} + \mathbf{H}_d, \qquad (1)$$

where $\mathbf{H}_0$ is the external field, λ the constant of the molecular field, H_{ex} the magnitude of the molecular field, l the interatomic distance, and $\mathbf{H}_d$ the demagnetizing field.

If we seek a solution of (1) in the linear approximation in the form of magnetization waves (spin waves) with the frequency ω_k and the wave vector $\mathbf{k}$, then the waves of wavelengths smaller than the dimensions of the system* satisfy the following dispersion relation

$$\omega_k = \gamma[(H_i + H_{ex}l^2k^2)(H_i + H_{ex}l^2k^2 + 4\pi M\sin^2\theta)]^{1/2}, \qquad (2)$$

where $H_i = H_0 - 4\pi M N_z$, $N_x = N_y$ and N_z are demagnetizing factors, the field H_0 is along the z axis, and θ is the angle between $\mathbf{k}$ and $\mathbf{H}_0$. The nonequilibrium state will be unstable with respect to spin waves if the following relation is satisfied

$$-4\pi M < H_i + H_{ex}l^2k^2 < 0. \qquad (3)$$

(In this case the frequencies ω_k become imaginary.)

For magnetostatic types of oscillation (for small $\mathbf{k}$)[1] we must replace $\sin^2\theta$ in (2) by

$$\beta_{nmr} = \delta^2(1 - z_{nmr}^2)/\delta^2(1 - z_{nmr}^2) + z_{nmr}^2,$$

where δ is the ratio between the major and the minor axes of the ellipsoid, and z_{nmr}^2 is the square of the r-th root of the associated Legendre polynomial $P_n^m(z)$ with $0 < z_{nmr}^2 < 1$.

We must take **k** = (n, m, r) and eliminate the terms $H_{ex}l^2k^2$.† It is clear that the oscillations begin to build up when $H_i < 0$.

Formula (2) is correct if the angle between $\mathbf{H}_0$ and **M** is close to 0 or π; however, calculations show that instability appears always whenever φ, the angle between $\mathbf{H}_0$ and **M**, is greater than $\pi/2$, since the spectrum for an arbitrary φ (but not close to $\pi/2$) has the form (for a spherical sample)

$$\omega_k = \gamma\left[\left(H' - \tfrac{1}{2}4\pi M\cos\varphi\sin^2\theta\right)\right.$$

$$\left.\times\left(H' + \tfrac{1}{2}4\pi M\cos\varphi\sin^2\theta\right)\right]^{1/2},$$

$$H' = H_0 + \left(\cos\varphi + \tfrac{1}{2}\tan\varphi\sin\varphi\right)\left(H_{ex}l^2k^2 - \tfrac{4}{3}\pi M\right)$$

$$+ \tfrac{1}{2}\cos\varphi 4\pi M\sin^2\theta + \tfrac{1}{2}4\pi M\sin\varphi\tan\varphi\cos^2\theta.$$

From the above formula it is clear that it is impossible to have a stable uniform precession of the magnetization of a ferromagnet if the angle between **H** and **M** is obtuse. In this sense the result of reference 2 on coherent radiation from inverted systems is incorrect when applied to ferromagnets. The same is true of the paper of Morgenthaler,[3] in which the question of ferromagnetic systems (ferrites) especially is discussed.

We notice that the characteristic time of occurrence of spin waves is of the order of $\tau = 1/4\pi\gamma M$.

2. For a ferrimagnet near the compensation point we can find a dispersion relation by using equations analogous to (1) (but with two sublattices). Analysis of this relation shows that the spin waves with $\theta = 0$ build up the fastest; for them

$$\omega_k = \gamma H_0 \pm [2\omega_{ex}K + K^2 + \omega_{ex}^2 l^2k^2]^{1/2},$$

$$K = \tfrac{1}{2}(\gamma_1 - \gamma_2)H_0 + \tfrac{1}{2}(\gamma_1 + \gamma_2)H_a; \qquad \omega_{ex} \approx \gamma H_{ex}.$$

The instability occurs when

$$2\omega_{ex}K + K^2 + \omega^2_{ex}l^2k^2 < 0. \tag{4}$$

Here γ_1 and γ_2 are the gyromagnetic ratios for the first and the second sublattices, and H_a the effective anisotropy field, which is the same for both sublattices.

We note that in the case of a ferrimagnet at the compensation point the frequency can become complex if condition (4) is fulfilled and k = 0, i.e., for uniform precession (with a build-up time smaller than for spin waves). This means that uniform precession predominates in the linear approximation considered here. In reference 4 account is taken of nonlinearity for uniform precession.

3. Suhl[5] has shown that there exists a nonlinear mechanism for the build-up of spin waves as a result of their interaction with the uniform precession. It was shown that for small angles φ, instability appeared if $\sin^2\varphi > \eta/4\pi M\gamma$, where η is the coefficient of relaxation of the spin waves. The characteristic instability time is $\tau \approx (4\pi M\gamma \times \sin^2\varphi - \eta)^{-1}$.

A calculation for a ferromagnet shows that for an arbitrary angle (but not close to $\pi/2$), instability occurs, the spin waves build up exponentially, and $\tau \approx 1/4\pi M\gamma\sin\varphi\tan\varphi$. In making this conclusion it is assumed that there already exists a uniform precession at an angle φ (in general not small) and we investigate its instability. The question considered above will be discussed in the journal Изв. высш. шк., Радиофизика (News of the Colleges, Radiophysics).

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*Rough calculations show that the dispersion relation (2) is correct for waves with wavelength ten times shorter than the dimensions of the sample.

†We note that the dependence of the magnetization of the magnetostatic oscillations on the coordinates is not described by the function $e^{i\mathbf{k}\cdot\mathbf{r}}$

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THEORY OF THE NONRESONANCE ABSORPTION OF AN OSCILLATING MAGNETIC FIELD BY A FERROMAGNETIC DIELECTRIC

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In a recently published article[1] we solved the problem of calculating the absorption coefficients of an oscillating magnetic field polarized in a direction perpendicular to the axis of easiest mag-

netization. Initially we had started with the Hamiltonian of the system which included only the exchange interaction.

Subsequently one of us (Tsukernik) showed that in this case there occurs only resonance absorption of a homogeneous oscillating magnetic field, since the total magnetic moment commutes with the Hamiltonian of the system. The results obtained in reference 1 are, thus, incorrect.

The erroneous result is due to the fact that in calculating the matrix transition elements we had confined ourselves to the first perturbation-theory approximation. Account of the second approximation shows that the matrix element becomes, with corresponding precision, zero.*

Nonresonance absorption of a homogeneous field is connected with relativistic interactions within the system (dipole-dipole interactions, the energy of anisotropy, etc.). This problem is studied in detail in an article which will be published later.

Here we will only cite the value of the absorption coefficient of a transverse magnetic field whose frequency is considerably greater than the frequency of the spin wave with a zero quasimomentum:

$$\Gamma_{\perp} \approx \frac{32\,\pi^2}{15\sqrt{2}}\, gM_0\, \frac{w}{\Theta_c}\, \frac{\mu\, M_0}{\sqrt{\hbar\omega\Theta_c}}\, \coth\, \frac{\hbar\omega}{4T}\,,$$

where $w = \mu^2/a^3$ (the notation is the same as in reference 1).

The absorption described by this formula is caused by the dissociation of a photon into two spin waves with opposite quasimomenta.

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